NORMAL FORMS FOR SEMILINEAR QUANTUM HARMONIC OSCILLATORS

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Abstract. — We consider the semilinear harmonic oscillator

$$i\psi_t = (-\Delta + |x|^2 + M)\psi + \partial_2 g(\psi, \bar{\psi}), \quad x \in \mathbb{R}^d, \ t \in \mathbb{R}$$

where M is a Hermite multiplier and g a smooth function globally of order 3 at least.
We prove that such a Hamiltonian equation admits, in a neighborhood of the origin, a Birkhoff normal form at any order and that, under generic conditions on M related to the non resonance of the linear part, this normal form is integrable when $d = 1$ and gives rise to simple (in particular bounded) dynamics when $d \geq 2$.
As a consequence we prove the almost global existence for solutions of the above equation with small Cauchy data. Furthermore we control the high Sobolev norms of these solutions.

***Résumé* (Formes normales de Birkhoff pour l'oscillateur harmonique quantique non linéaire)**

Dans cet article nous considérons l'oscillateur harmonique semi-linéaire :

$$i\psi_t = (-\Delta + x^2 + M)\psi + \partial_2 g(\psi, \bar{\psi}), \quad x \in \mathbb{R}^d, \ t \in \mathbb{R}$$

où M est un multiplicateur de Hermite et g est une fonction régulière globalement d'ordre au moins trois.
Nous montrons qu'une telle équation admet, au voisinage de zéro, une forme normale de Birkhoff à n'importe quel ordre et que, sous des hypothèses génériques sur M liées à la non résonance de la partie linéaire, cette forme normale est complètement intégrable si $d = 1$ et donne lieu à une dynamique simple (et en particulier bornée) pour $d \geq 2$.
Ce résultat nous permet de démontrer l'existence presque globale et de contrôler les normes de Sobolev d'indice grand des solutions de l'équation non linéaire ci-dessus avec donnée initiale petite.

Key words and phrases. — Birkhoff normal form, Semilinear quantum harmonic oscillator, Hamiltonian PDEs, long time stability, Gross-Pitaevskii equation. AMS classification: 37K55,37K45, 35B34, 35B35.

1. Introduction, statement of the results

The aim of this paper is to prove a Birkhoff normal form theorem for the semilinear harmonic oscillator equation

$$\begin{cases} i\psi_t = (-\Delta + |x|^2 + M)\psi + \partial_2 g(\psi,\bar\psi) \\ \psi|_{t=0} = \psi_0 \end{cases} \tag{1.1}$$

on the whole space $\mathbb{R}^d$ ($d \geq 1$) and to discuss its dynamical consequences. Here g is a smooth function, globally of order $p \geq 3$ at 0, and $\partial_2 g$ denotes the partial derivative of g with respect to the second variable. The linear operator M is a Hermite multiplier. To define it precisely (at least in the case $d = 1$, see Section 3.2 for the multidimensional case), let us introduce the quantum harmonic oscillator on $\mathbb{R}^d$, denoted by $T = -\Delta + |x|^2$. When $d = 1$, T is diagonal in the Hermite basis $(\phi_j)_{j\in\bar{\mathbb{N}}}$:

$$T\phi_j = (2j-1)\phi_j, \quad j \in \bar{\mathbb{N}}$$
$$\phi_{n+1} = \frac{H_n(x)}{\sqrt{2^n n!}} e^{-x^2/2}, \quad n \in \mathbb{N}$$

where $H_n(x)$ is the n^{th} Hermite polynomial relative to the weight e^{-x^2} :

$$\int_{\mathbb{R}} e^{-x^2} H_m(x) H_n(x) dx = 2^n n! \sqrt{\pi}\,\delta_{nm}.$$

In this basis (and for $d = 1$), a Hermite multiplier is an operator given by

$$M\phi_j = m_j \phi_j\,,$$

where $(m_j)_{j\in\bar{\mathbb{N}}}$ is a bounded sequence of real numbers, that will be chosen in the following classes : for any $k \geq 1$, we define the class

$$\mathcal{W}_k = \{(m_j)_{j\in\bar{\mathbb{N}}} \mid \text{ for each } j,\ m_j = \frac{\tilde m_j}{j^k} \text{ with } \tilde m_j \in [-1/2, 1/2]\}$$

that we endow with the product probability measure. In this context the linear frequencies, i.e. the eigenvalues of $T + M = -d^2/dx^2 + x^2 + M$ are given by

$$\omega_j = 2j - 1 + m_j = 2j - 1 + \frac{\tilde m_j}{j^k}, \quad j \in \bar{\mathbb{N}}. \tag{1.2}$$

Let

$$\begin{aligned} \tilde H^s = \{ f \in H^s(\mathbb{R}^d, \mathbb{C}) | x \mapsto x^\alpha \partial^\beta f \in L^2(\mathbb{R}^d) \\ \text{for any } \alpha,\ \beta \in \mathbb{N}^d \text{ satisfying } 0 \leq |\alpha| + |\beta| \leq s \} \end{aligned} \tag{1.3}$$

where $H^s(\mathbb{R}^d, \mathbb{C})$ is the standard Sobolev space on $\mathbb{R}^d$. We note that, for any $s \geq 0$, the domain of $T^{s/2}$ is $\tilde H^s$ (see for instance [**Hel84**] Proposition 1.6.6)

and that for $s > d/2$, $\tilde{H}^s$ is an algebra.
If $\psi_0 \in \tilde{H}^s$ is small, say of norm ϵ, local existence theory implies that (1.1) admits a unique solution in $\tilde{H}^s$ defined on an interval of length $c\epsilon^{-p+2}$. Our goal is to prove that for M outside an exceptional subset, given any integer $r \geq 1$ and provided that s is large enough and ϵ is small enough, the solution extends over an interval of length $c\epsilon^{-r}$. Furthermore we control the norm of the solution in $\tilde{H}^s$-norm ($d \geq 1$) and localize the solution in the neighborhood of a torus (only in the case $d = 1$, cf. Theorem 3.4 and Theorem 3.10).
Precisely we have

Theorem 1.1. — *Let $r, k \in \mathbb{N}$ be arbitrary integers. There exists a set $F_k \subset \mathcal{W}_k$ whose measure equals 1 such that if $m = (m_j)_{j\in\bar{\mathbb{N}}} \in F_k$ and if g is a C^∞ function on a neighborhood of the origin in $\mathbb{C}^2$, satisfying $g(z,\bar{z}) \in \mathbb{R}$ and vanishing at least at order 3 at the origin, there is $s_0 \in \mathbb{N}$ such that for any $s \geq s_0$, f there are $\epsilon_0 > 0$, $c > 0$, such that for any $\epsilon \in (0, \epsilon_0)$, for any ψ_0 in $\tilde{H}^s$ with $||\psi_0||_s \leq \epsilon$, the Cauchy problem (1.1) with initial datum ψ_0 has a unique solution*

$$\psi \in C^1((-T_\epsilon, T_\epsilon), \tilde{H}^s)$$

with $T_\epsilon \geq c\epsilon^{-r}$. Moreover, for any $t \in (-T_\epsilon, T_\epsilon)$, one has

$$\|\psi(t,\cdot)\|_{\tilde{H}^s} \leq 2\epsilon\ . \tag{1.4}$$

For the nonlinearity $g(\psi,\bar{\psi}) = \lambda\frac{2}{p+1}|\psi|^{p+1}$ with $p \geq 1$ and without Hermite multiplier ($M = 0$), we recover the Gross-Pitaevskii equation

$$i\psi_t = (-\Delta + |x|^2)\psi + \lambda|\psi|^{p-1}\psi, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^d. \tag{1.5}$$

In this case, the global existence in the energy space $\tilde{H}^1$ has been proved for[(1)] $1 \leq p < \frac{d+2}{(d-2)^+}$ without smallness assumption on the Cauchy data in the defocusing case ($\lambda < 0$) and for small Cauchy data in the focusing case ($\lambda > 0$) (see [**Car02**] and also [**Zha05**]). But nothing is known for nonlinearities of higher order, neither about conservation of the $\tilde{H}^s$-norm for $s > 1$. Our result states that, avoiding resonances by adding a generic linear term $M\psi$ (but $M = 0$ is not allowed), we recover almost global existence for solutions of Gross-Pitaevskii equation with a nonlinearity of arbitrary high order and small Cauchy data in $\tilde{H}^s$ for s large enough. In some sense, this shows that the instability for Gross-Pitaevskii that could appear in that regime are necessarily produced by resonances. More precisely, we can compare with

(1) we use the convention $\frac{d+2}{(d-2)^+} = +\infty$ for $d = 1, 2$, and $(d-2)^+ = d - 2$ for $d \geq 3$

the semi-classical cubic Gross-Pitaevskii in $\mathbb{R}^3$ which appears in the study of Bose-Einstein condensates (for a physical presentation see [**PS03**])

$$ihu_t = -h^2\Delta u + |x|^2 u + h^2|u|^2 u, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^3 \tag{1.6}$$

where h is a small parameter.
The scaling relation between ψ solution of (1.5) and u solution of (1.6) is given by

$$u(t,x) = \frac{1}{\sqrt{h}}\psi(t, \frac{x}{\sqrt{h}}). \tag{1.7}$$

We note that for multi indices $\alpha, \beta \in \mathbb{N}^3$, with $y = \frac{x}{\sqrt{h}}$,

$$\left\| y^\alpha \partial^\beta \psi \right\|^2_{L^2(\mathbb{R}^3)} = h^{|\beta|-|\alpha|-1/2} \left\| x^\alpha \partial^\beta u \right\|^2_{L^2(\mathbb{R}^3)}.$$

Thus the smallness of ψ_0 in $\tilde{H}^s$ imposed in Theorem 1.1, i.e. $\|\psi_0\|_{\tilde{H}^s} \leq C\epsilon$, reads

$$\sum_{|\beta|+|\alpha|\leq s} h^{|\beta|-|\alpha|-1/2} \left\| x^\alpha \partial^\beta u_0 \right\|^2_{L^2(\mathbb{R}^3)} \leq C\epsilon^2.$$

Taking $\epsilon = h^{1/6}$ with h small enough, this allows the derivatives of order greater than 1 to have large L^2-norm when h is small:

$$\left\| \partial^\beta u_0 \right\|^2_{L^2(\mathbb{R}^3)} = O(h^{-|\beta|+5/6})$$

i.e. the initial data has to be small in L^2 but may have large oscillations. Then, Theorem 1.1 states that, avoiding the resonances by adding a generic linear term (which, in the preceding scaling, stays of order h), the same estimates remain true for the solution $u(t,.)$ with $|t| = O(h^{-r/6})$, r being chosen arbitrarily from the principle. Notice that the role of the linear operator M is to remove the resonances between the free modes (see (1.2)). The fully resonant case $M = 0$ is beyond the scope of the paper.

To prove Theorem 1.1 we use the Birkhoff normal form theory. This technique has been developped by Bourgain [**Bou96**], Bambusi [**Bam03**], Bambusi-Grébert [**BG06**] for semilinear PDEs (typically semilinear Schrödinger equation or semilinear wave equation) on the one dimensional torus and by Bambusi-Delort-Grébert-Szeftel [**BDGS07**] for the semilinear Klein-Gordon equation on the sphere S^d (or a Zoll manifold). These cases were concerned with compact domains. In our work the domain is $\mathbb{R}^d$, the potential x^2 guarantees that the spectrum remains pure point, but the free modes of the harmonic oscillator are not so well localized.

For general reference on Hamiltonian PDEs and their perturbations, see the recent monographies [**Cra00, Kuk00, Bou05, KP03**]. We also note that in [**Kuk93**], a KAM-like theorem is proved for (1.1) in one dimension and with special nonlinearities .

Let us describe roughly the general method. Consider a Hamiltonian system whose Hamiltonian function decomposes in a quadratic part, H_0 (associated to the linear part of the equation), and a perturbative nonlinear part P (at least cubic): $H = H_0 + P$. We assume that H_0 is diagonal in a Hilbert basis $(\phi_j)_{j\geq 1}$ of the phase space $\mathcal{P}$: $H_0 = \sum_j \omega_j \xi_j \eta_j$ for $(\xi, \eta) \in \mathcal{P}$ and $\omega = (\omega_j)_{j\geq 1}$ is the vector of free frequencies (the eigenvalues of the linear part). In the harmonic oscillator case, the Hilbert basis is given by the Hermite functions and $\mathcal{P} = \ell^2 \times \ell^2$. The heuristic idea could be resumed as follows: if the free modes do not interact linearly (i.e. if ω is non resonant), and if they do not interact too much via the nonlinear term, then the system will remain close to an integrable one, up to a nonlinear term of very high order, and thus the solutions will exist and stay under control during a very long time. More precisely, by a Birkhoff normal form approach we prove (cf. Theorem 2.23 which is our main theorem) that $H \sim H_0' + P'$ where H_0' is no more quadratic but remains integrable (in the case $d = 1$) and P' is at least of order r, where r can be chosen arbitrarily large as soon as we work in a sufficiently small neighborhood of the origin.

To guarantee the second condition, i.e. that the free modes do not interact too much via the nonlinear term, we have to control the integral of the product of three or more modes:

$$a_j = \int_D \phi_{j_1}(x) \cdots \phi_{j_k}(x) dx \tag{1.8}$$

where D is the space domain ($\mathbb{R}^d$ in our case) and j is a multi-index in $\mathbb{N}^k$, k being smaller than the fixed order r and larger than 3. It turns out that, in our case, this control cannot be as good as in the cases of compact domains studied previously.

Let us consider ordered multi-indices j, i.e. such that $j_1 \geq j_2 \geq \cdots \geq j_k$. In [**BDGS07, Gré07, Bam07**] the following control was used: there exists $\nu > 0$ and for any $N \geq 1$ there exists $C_N > 0$ such that for all ordered j

$$|a_j| \leq C_N j_3^{\nu} \left(\frac{j_3}{j_3 + j_1 - j_2} \right)^N . \tag{1.9}$$

In the case of the harmonic oscillator, this estimate is false (cf [**Wan08**] where an equivalent is computed for four modes) and we are only able to prove the following: there exists $\nu > 0$ and for any $N \geq 1$ there exists $C_N > 0$ such that for all ordered j

$$|a_j| \leq C_N \frac{j_3^\nu}{j_1^{1/24}} \left(\frac{\sqrt{j_2 j_3}}{\sqrt{j_2 j_3} + j_1 - j_2} \right)^N . \tag{1.10}$$

The difference could seem minimal but it is technically important: $\sum_{j_1} \left(\frac{j_3}{j_3 + j_1 - j_2} \right)^\mu \sim C j_3$ for an uniform constant C providing $\mu > 1$ and similarly $\sum_{j_1} \left(\frac{\sqrt{j_2 j_3}}{\sqrt{j_2 j_3} + j_1 - j_2} \right)^\mu \sim C\sqrt{j_2 j_3}$ for $\mu > 1$. In the first case, the extra term j_3 can be absorbed by changing the value of ν in (1.9) ($\nu' = \nu + 1$). This is not possible in the second case. In some sense the perturbative nonlinearity is no longer short range (cf. [**Wan08**]).

Actually in the case studied in [**Bou96, Bam03, BG06**], the linear modes (i.e. the eigenfunctions of the linear part) are localized around the exponentials e^{ikx}, i.e. the eigenfunctions of the Laplacian on the torus. In particular the product of eigenfunctions is close to an other eigenfunction which makes the control of (1.8) simpler. In the harmonic oscillator case, the eigenfunctions are not localized and the product of eigenfunctions has more complicated properties. Notice that, in the case of the semilinear Klein-Gordon equation on the sphere, the control of (1.8) is more complicated to obtain, but an estimate of type (1.9) is proved in [**DS04**] for the Klein-Gordon equation on Zoll manifolds.

From the point of view of a normal form, the substitution of (1.9) by (1.10) has the following consequence:

Consider a formal polynomial

$$Q(\xi, \eta) \equiv Q(z) = \sum_{l=0}^{k} \sum_{j \in \mathbb{N}^l} a_j z_{j_1} \dots z_{j_l}$$

with coefficients a_j satisfying (1.9). In [**Gré07**] or [**Bam07**], it is proved that its Hamiltonian vector field X_Q is then regular from[(2)] $\mathcal{P}_s = \ell_s^2 \times \ell_s^2$ to $\mathcal{P}_s$ for all s large enough (depending on ν). In our present case, i.e. if a_j only satisfy (1.10), which defines the class $\mathcal{T}^\nu$, then we prove that X_Q is regular from $\mathcal{P}_s$ to $\mathcal{P}_{s'}$ for all $s' < s - 1/2 + 1/24$ and s large enough. This "loss of regularity" would of course complicate an iterative procedure, but it is bypassed in the following way : the nonlinearity P is regular in the sense that X_P maps $\mathcal{P}_s$ to $\mathcal{P}_s$ continuously for s large enough (essentially because the space $\tilde{H}^s$ is an algebra for $s > d/2$). On the other hand, we build at each step a

(2) here $l_s^2 = \{(z_l) \mid \sum l^{2s} |z_l|^2 < \infty\}$ and corresponds to functions $\psi = \sum z_l \phi_l$ in $\tilde{H}^{2s}$.

canonical transform which preserves the regularity. Indeed, at each iteration, we compute the canonical transformation as the time 1 flow of a Hamiltonian χ, and the solution of the so called homological equation gives rise to an extra term in (1.10) for the coefficient of the polynomial χ:

$$|a_j| \leq C_N \frac{j_3^{\nu}}{j_1^{1/24}(1+j_1-j_2)} \left(\frac{\sqrt{j_2 j_3}}{\sqrt{j_2 j_3}+j_1-j_2} \right)^N . \tag{1.11}$$

Using such an estimate on the coefficients (in the class[3] denoted $\mathcal{T}^{\nu,+}$ in Section 2.2), we prove in Proposition 2.13 that X_χ is regular from $\mathcal{P}_s$ to $\mathcal{P}_s$ for all s large enough. Furthermore, we prove in Proposition 2.18 that the Poisson bracket of a polynomial in $\mathcal{T}^\nu$ with a polynomial in $\mathcal{T}^{\nu,+}$ is in $\mathcal{T}^{\nu'}$ for some ν' larger than ν. So an iterative procedure is possible in $\mathcal{P}_s$.
This smoothing effect of the homological equations was already used by S. Kuksin in [**Kuk87**] (see also [**Kuk93, Pös96**]). Notice that this is, in some sense, similar to the local smoothing property for Schrödinger equations with potentials superquadratic at infinity studied in [**YZ04**].

Our article is organized as follows: in Section 2 we state and prove a specific Birkhoff normal form theorem adapted to the loss of regularity that we explained above. In Section 3, we apply this theorem to the $1-d$ semilinear harmonic oscillator equation (Subsection 3.1) and we generalize it to cover the multidimensional case (Subsection 3.2).

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2. The Birkhoff normal form

2.1. The abstract model. — To begin with, we give an abstract model of infinite dimensional Hamiltonian system. In Section 3 we will verify that the nonlinear harmonic oscillator can be described in this abstract framework. Throughout the paper, we denote $\bar{\mathbb{N}} = \mathbb{N} \setminus \{0\}$ and $\bar{\mathbb{Z}} = \mathbb{Z} \setminus \{0\}$. We work in the phase space $\mathcal{P}_s \equiv \mathcal{P}_s(\mathbb{C}) := \ell^2_s(\mathbb{C}) \times \ell^2_s(\mathbb{C})$ where, for $s \in \mathbb{R}_+$, $\ell^2_s(\mathbb{C}) := \{(a_j)_{j\geq 1} \in \mathbb{C}^{\bar{\mathbb{N}}} \mid \sum_{j\geq 1} j^{2s}|a_j|^2 < +\infty\}$ is a Hilbert space for the standard norm: $\|a\|_s^2 = \sum_{j\geq 1} |j|^{2s}|a_j|^2$. We denote $\mathcal{P}_s(\mathbb{R}) := \{(\xi,\bar{\xi}) \in \mathcal{P}_s(\mathbb{C})\}$ the "real" part of $\mathcal{P}_s(\mathbb{C})$. We shall denote a general point of $\mathcal{P}_s$ by $z = (\xi, \eta)$ with $z = (z_j)_{j\in\bar{\mathbb{Z}}}$,

[3] Actually in section 2.2, instead of $\mathcal{T}^\nu$ and $\mathcal{T}^{\nu,+}$, we consider more general classes $\mathcal{T}^{\nu,\beta}$ and $\mathcal{T}^{\nu,\beta,+}$ where the parameter β plays the role of the exponent $1/24$ in (1.10) and (1.11)

$\xi = (\xi_j)_{j\in\bar{\mathbb{N}}}$, $\eta = (\eta_j)_{j\in\bar{\mathbb{N}}}$ and the correspondence: $z_j = \xi_j$, $z_{-j} = \eta_j$ for all $j \in \bar{\mathbb{N}}$. Finally, for a Hamiltonian function H, the Hamiltonian vector field X_H is defined by

$$X_H(z) = \left(\left(-\frac{\partial H}{\partial \xi_k}\right)_{k\in\bar{\mathbb{N}}}, \left(\frac{\partial H}{\partial \eta_k}\right)_{k\in\bar{\mathbb{N}}}\right).$$

Definition 2.1. — *Let $s \geq 0$, we denote by $\mathcal{H}^s$ the space of Hamiltonian functions H defined on a neighborhood $\mathcal{U}$ of the origin in $\mathcal{P}_s \equiv \mathcal{P}_s(\mathbb{C})$, satisfying $H(\xi,\bar{\xi}) \in \mathbb{R}$ (we say that H is real) and*

$$H \in C^\infty(\mathcal{U},\mathbb{C}) \quad \text{and} \quad X_H \in C^\infty(\mathcal{U},\mathcal{P}_s),$$

as well as every homogeneous polynomial H_k appearing in the Taylor expansion of H at 0 :

$$H_k \in C^\infty(\mathcal{U},\mathbb{C}) \quad \text{and} \quad X_{H_k} \in C^\infty(\mathcal{U},\mathcal{P}_s).$$

Remark 2.2. — This property, for Hamiltonians contributing to the nonlinearity, will in particular force them to be *semilinear perturbations* of the harmonic oscillator.

In particular the Hamiltonian vector fields of functions F, G in $\mathcal{H}^s$ are in $\ell^2_s(\mathbb{C}) \times \ell^2_s(\mathbb{C})$ and we can define their Poisson bracket by

$$\{F,G\} = i\sum_{j\geq 1} \frac{\partial F}{\partial \xi_j}\frac{\partial G}{\partial \eta_j} - \frac{\partial F}{\partial \eta_j}\frac{\partial G}{\partial \xi_j} \ .$$

Notice that since for $P \in \mathcal{H}^s$, the vector field X_P is a C^∞ function from a neighborhood of $\mathcal{P}_s$ to $\mathcal{P}_s$ we have

Lemma 2.3. — *Let $P \in \mathcal{H}^s$ such that P vanishes up to order $r+1$ at the origin, that is :*

$$\forall k \leq r+1, \forall j \in \bar{\mathbb{Z}}^k, \ \frac{\partial^k P}{\partial z_{j_1}\ldots\partial z_{j_k}}(0) = 0$$

Then there exists $\varepsilon_0 > 0$ and $C > 0$ such that, for $z \in \mathcal{P}_s$ satisfying $||z||_s \leq \varepsilon_0$, we have

$$||X_P(z)||_s \leq C||z||_s^r.$$

Our model of integrable system is the harmonic oscillator

$$H_0 = \sum_{j\geq 1}\omega_j\xi_j\eta_j$$

where $\omega = (\omega_j)_{j\geq 1} \in \mathbb{R}^{\bar{\mathbb{N}}}$ is the frequency vector. We will assume that these frequencies grow at most polynomially, i.e. that there exist $C > 0$ and $\bar{d} \geq 0$ such that for any $j \in \bar{\mathbb{N}}$,

$$|\omega_j| \leq C|j|^{\bar{d}}, \tag{2.1}$$

in such a way that H_0 be well defined on $\mathcal{P}_s$ for s large enough.
The perturbation term is a real function, $P \in \mathcal{H}^s$, having a zero of order at least 3 at the origin. Our Hamiltonian function is then given by

$$H = H_0 + P$$

and Hamilton's canonical equations read

$$\begin{cases} \dot{\xi}_j &= -i\omega_j\xi_j - i\frac{\partial P}{\partial \eta_j},\ j \geq 1 \\ \dot{\eta}_j &= i\omega_j\eta_j + i\frac{\partial P}{\partial \xi_j},\ j \geq 1. \end{cases} \tag{2.2}$$

Our theorem will require essentially two hypotheses: one on the perturbation P (see Definition 2.6) and one on the frequency vector ω that we describe now.

For $j \in \bar{\mathbb{Z}}^k$ with $k \geq 3$, we define $\mu(j)$ as the third largest integer among $|j_1|, \dots, |j_k|$. Then we set $S(j) := |j_{i_1}| - |j_{i_2}|$ where $|j_{i_1}|$ and $|j_{i_2}|$ are respectively the largest integer and the second largest integer among $|j_1|, \dots, |j_k|$. In particular, if the multi-index j is ordered i.e. if $|j_1| \geq \ldots \geq |j_k|$ then

$$\mu(j) := |j_3| \text{ and } S(j) = |j_1| - |j_2|.$$

In [**Bam03, BG06, Gré07, Bam07**] the non resonance condition on ω reads

Definition 2.4. — *A frequency vector $\omega \in \mathbb{R}^{\bar{\mathbb{N}}}$ is* ***non resonant*** *if for any $r \in \bar{\mathbb{N}}$, there are $\gamma > 0$ and $\delta > 0$ such that for any $j \in \bar{\mathbb{N}}^r$ and any $1 \leq i \leq r$, one has*

$$\left|\omega_{j_1} + \dots + \omega_{j_i} - \omega_{j_{i+1}} - \dots - \omega_{j_r}\right| \geq \frac{\gamma}{\mu(j)^\delta} \tag{2.3}$$

except in the case $\{j_1, \dots, j_i\} = \{j_{i+1}, \dots, j_r\}$.

In the harmonic oscillator case[4], we are able to work with a slightly refined non resonance condition

[4] The following holds, more generally, if the frequency vector is non resonant as in Definition 2.4 and satisfies the asymptotic: $\omega_l \sim l^n$ with $n \geq 1$.

Definition 2.5. — *A frequency vector $\omega \in \mathbb{R}^{\bar{\mathbb{N}}}$ is* ***strongly non resonant*** *if for any $r \in \bar{\mathbb{N}}$, there are $\gamma > 0$ and $\delta > 0$ such that for any $j \in \bar{\mathbb{N}}^r$ and any $1 \leq i \leq r$, one has*

$$\left|\omega_{j_1} + \cdots + \omega_{j_i} - \omega_{j_{i+1}} - \cdots - \omega_{j_r}\right| \geq \gamma \frac{1 + S(j)}{\mu(j)^\delta} \tag{2.4}$$

except if $\{j_1, \ldots, j_i\} = \{j_{i+1}, \ldots, j_r\}$.

This improvement of the non resonance condition is similar to the modification to the standard second Melnikov condition introduced first by S. Kuksin in [**Kuk87**] (see also [**Kuk93**] and [**Pös96**]).

2.2. Polynomial structure. — For $j \in \bar{\mathbb{Z}}^k$ with $k \geq 3$, we have already defined $\mu(j)$ and $S(j)$, we now introduce

$$B(j) = |j_{i_2}\ j_{i_3}|^{1/2}, \quad C(j) = |j_{i_1}|$$

where $|j_{i_1}|$, $|j_{i_2}|$ and $|j_{i_3}|$ are respectively the first, the second and the third largest integer among $|j_1|, \ldots, |j_k|$. We also define

$$A(j) = \frac{B(j)}{B(j) + S(j)}. \tag{2.5}$$

In particular, if the multi-index j is ordered i.e. if $|j_1| \geq \ldots \geq |j_k|$ then

$$A(j) = \frac{|j_2 j_3|^{1/2}}{|j_2 j_3|^{1/2} + |j_1| - |j_2|}$$

and

$$C(j) = |j_1|.$$

Definition 2.6. — *Let $k \geq 3$, $\beta \in (0, +\infty)$ and $\nu \in [0, +\infty)$ and let*

$$Q(\xi, \eta) \equiv Q(z) = \sum_{j \in \bar{\mathbb{Z}}^k} a_j z_{j_1} \ldots z_{j_l} \tag{2.6}$$

be a formal homogeneous polynomial of degree k on $\mathcal{P}_s(\mathbb{C})$. Q is in the class $\mathcal{T}_k^{\nu,\beta}$ if for any $N \geq 1$ there exists a constant $c_N > 0$ such that for all $j \in \bar{\mathbb{Z}}^k$

$$|a_j| \leq c_N \frac{\mu(j)^\nu}{C(j)^\beta} A(j)^N. \tag{2.7}$$

We will also use

Definition 2.7. — *Let $k \geq 3$, $\beta \in [0, +\infty)$ and $\nu \in [0, +\infty)$ and let*

$$Q(\xi, \eta) \equiv Q(z) = \sum_{j \in \bar{\mathbb{Z}}^k} a_j z_{j_1} \ldots z_{j_l}$$

be a formal homogeneous polynomial of degree k on $\mathcal{P}_s(\mathbb{C})$. Q is in the class $\mathcal{T}_k^{\nu,\beta,+}$ if for any $N \geq 1$ there exists a constant $c_N > 0$ such that for all $j \in \bar{\mathbb{Z}}^k$

$$|a_j| \leq c_N \frac{\mu(j)^\nu}{C(j)^\beta(1+S(j))} A(j)^N. \tag{2.8}$$

The best constants c_N in (2.7) define a family of semi-norms for which $\mathcal{T}_k^{\nu,\beta}$ is a Fréchet space.

Remark 2.8. — Notice that the formula (2.6) does not give a unique representation of polynomials on $\mathcal{P}_s$. However, since the estimates (2.7) and (2.8) are symmetric with respect to the order of the indexes $j_1, \cdots, j_k$, this non uniqueness does not affect Definitions 2.6 and 2.7.

Remark 2.9. — In the estimate (2.7), the numerator allows an increasing behaviour with respect to $\mu(j)$ that will be useful to control the small divisors. The denominator imposes a slightly decreasing behaviour with respect to the largest index $C(j)$ and a highly decreasing behaviour for monomials having their two modes of largest indexes that are not of the same order. This control is slightly better in $\mathcal{T}_k^{\nu,\beta,+}$.

Remark 2.10. — We will see in Proposition 2.13 that, if $\beta > 1/2$ then $\mathcal{T}_k^{\nu,\beta} \subset \mathcal{H}^s$ for $s \geq \nu + 1$. Unfortunately β is not that large in the harmonic oscillator case, where the best we obtain is $\beta = 1/24$. Thus $P \in \mathcal{T}_k^{\nu,\beta}$ does not imply $P \in \mathcal{H}^s$. Nevertheless, as we will see in Proposition 2.13, a polynomial in $\mathcal{T}_k^{\nu,\beta}$ is well defined and continuous on a neighborhood of the origin in $\mathcal{P}_s(\mathbb{C})$ for s large enough. As a comparison, in [**Gré07, Bam07**], our estimate (2.7) is replaced with

$$|a_j| \leq C_N \frac{\mu(j)^{N+\nu}}{(\mu(j)+S(j))^N}. \tag{2.9}$$

which is actually better than (2.7), since it implies the $\mathcal{H}^s$ regularity. This type of control on the coefficients a_j was first introduced in [**DS04**] in the context of multilinear forms.

Definition 2.11. — *Let $\nu \geq 0$ and $\beta \geq 0$. A function P is in the class $\mathcal{T}^{\nu,\beta}$ if*

- *there exists $s_0 \geq 0$ such that, for any $s \geq s_0$ there exists $\mathcal{U}_s$, a neighborhood of the origin in $\mathcal{P}_s$ such that $P \in C^\infty(\mathcal{U}_s, \mathbb{C})$.*
- *P has a zero of order at least 3 in 0.*
- *for each $k \geq 3$ the Taylor's expansion of degree k of P at zero belongs to $\otimes_{l=3}^k \mathcal{T}_l^{\nu,\beta}$.*

We now define the class of polynomials in **normal form**:

Definition 2.12. — *Let $k = 2m$ be an even integer. A formal homogeneous polynomial Z of degree k on $\mathcal{P}_s$ is in normal form if it reads*

$$Z(z) = \sum_{j \in \bar{\mathbb{N}}^m} b_j z_{j_1} z_{-j_1} \dots z_{j_m} z_{-j_m} \tag{2.10}$$

i.e. Z depends only on the actions $I_l := z_l z_{-l} = \xi_l \eta_l$.

The aim of the Birkhoff normal form theorem is to reduce a given Hamiltonian of the form $H_0 + P$ with P in $\mathcal{H}^s$ to a Hamiltonian of the form $Z + R$ where Z is in normal form and R remains very small, in the sense that it has a zero of high order at the origin.

We now review the properties of polynomials in the class $\mathcal{T}^{\nu,\beta}$.

Proposition 2.13. — *Let $k \in \bar{\mathbb{N}}$, $\nu \in [0, +\infty)$, $\beta \in [0, +\infty)$, $s \in \mathbb{R}$ with $s > \nu + 1$, and let $P \in \mathcal{T}^{\nu,\beta}_{k+1}$. Then*

(i) *P extends as a continuous polynomial on $\mathcal{P}_s(\mathbb{C})$ and there exists a constant $C > 0$ such that for all $z \in \mathcal{P}_s(\mathbb{C})$*

$$|P(z)| \leq C \|z\|_s^{k+1}$$

(ii) *For any $s' < s + \beta - \frac{1}{2}$, the Hamiltonian vector field X_P extends as a bounded function from $\mathcal{P}_s(\mathbb{C})$ to $\mathcal{P}_{s'}(\mathbb{C})$. Furthermore, for any $s_0 \in (\nu + 1, s]$, there is $C > 0$ such that for any $z \in \mathcal{P}_s(\mathbb{C})$*

$$\|X_P(z)\|_{s'} \leq C \|z\|_s \|z\|_{s_0}^{(k-1)}. \tag{2.11}$$

(iii) *Assume moreover that $P \in \mathcal{T}^{\nu,\beta,+}_{k+1}$ with $\beta > 0$, then the Hamiltonian vector field X_P extends as a bounded function from $\mathcal{P}_s(\mathbb{C})$ to $\mathcal{P}_s(\mathbb{C})$. Furthermore, for any $s_0 \in (\nu + 1, s]$, there is $C > 0$ such that for any $z \in \mathcal{P}_s(\mathbb{C})$*

$$\|X_P(z)\|_s \leq C \|z\|_s \|z\|_{s_0}^{(k-1)}. \tag{2.12}$$

(iv) *Assume finally that $P \in \mathcal{T}^{\nu,\beta}_{k+1}$ and P is in* normal form *in the sense of Definition 2.12, then the Hamiltonian vector field X_P extends as a bounded function from $\mathcal{P}_s(\mathbb{C})$ to $\mathcal{P}_s(\mathbb{C})$. Furthermore, for any $s_0 \in (\nu, s]$, there is $C > 0$ such that for any $z \in \mathcal{P}_s(\mathbb{C})$*

$$\|X_P(z)\|_s \leq C \|z\|_s \|z\|_{s_0}^{(k-1)}. \tag{2.13}$$

Remark 2.14. — Since homogeneous polynomials are their own Taylor expansion at 0, assertions (iii) and (iv) imply that every element of $\mathcal{T}^{\nu,\beta,+}_{k+1}$, and every element of $\mathcal{T}^{\nu,\beta}_{k+1}$ in normal form is in $\mathcal{H}^s$.

Proof. — (i) Let P be an homogeneous polynomial of degree $k+1$ in $\mathcal{T}_{k+1}^{\nu,\beta}$ and for $z \in \mathcal{P}_s(\mathbb{C})$ write

$$P(z) = \sum_{j \in \bar{\mathbb{Z}}^{k+1}} a_j \, z_{j_1} \ldots z_{j_{k+1}} \,. \tag{2.14}$$

One has, using first (2.7) and then that $A(j) \leq 1$, $C(j) \geq 1$,

$$\begin{aligned}
|P(z)| &\leq C \sum_{j \in \bar{\mathbb{Z}}^{k+1}} \frac{\mu(j)^\nu}{C(j)^\beta} A(j)^N \prod_{i=1}^{k+1} |z_{j_i}| \\
&\leq C \sum_{j \in \bar{\mathbb{Z}}^{k+1}} \frac{\mu(j)^\nu}{\prod_{i=1}^{k+1} |j_i|^s} \prod_{i=1}^{k+1} |j_i|^s |z_{j_i}| \\
&\leq C \sum_{j \in \bar{\mathbb{Z}}^{k+1}} \frac{1}{\prod_{i=1}^{k+1} |j_i|^{s-\nu}} \prod_{i=1}^{k+1} |j_i|^s |z_{j_i}| \\
&\leq C \left(\sum_{l \in \bar{\mathbb{Z}}} \frac{1}{|l|^{2s-2\nu}} \right)^{\frac{k+1}{2}} \|z\|_s^{k+1}
\end{aligned}$$

where, in the last inequality, we used $k+1$ times the Cauchy-Schwarz inequality. Since $s > \nu + 1/2$, the last sum converges and the first assertion is proved.

(ii) We have to estimate the derivative of polynomial P with respect to any of its variables. Because of (2.7), given any N, we get

$$\left|\frac{\partial P}{\partial z_l}\right| \leq C_N (k+1) \sum_{j \in \bar{\mathbb{Z}}^k} \frac{\mu(j,l)^\nu}{C(j,l)^\beta} A(j,l)^N |z_{j_1}| \ldots |z_{j_k}| \,,$$

where the quantities $\mu(j,l)$, $C(j,l)$ and $A(j,l)$ are computed for the $k+1$-tuple made of $j_1,\dots,j_k,l$. Furthermore

$$\begin{aligned}
||X_P(z)||_{s'}^2 &\le C\sum_{l\in\bar{\mathbb{Z}}}\left(\sum_{j\in\bar{\mathbb{Z}}^k}\frac{|l|^{s'}\mu(j,l)^\nu}{C(j,l)^\beta}A(j,l)^N|z_{j_1}|\dots|z_{j_k}|\right)^2\\
&\le C(k!)^2\sum_{l\in\bar{\mathbb{Z}}}\left(\sum_{j\in\bar{\mathbb{Z}}^k_>}\frac{|l|^{s'}\mu(j,l)^\nu}{C(j,l)^\beta}A(j,l)^N|z_{j_1}|\dots|z_{j_k}|\right)^2\\
&\le C'||z||_{s_0}^{2(k-3)}\sum_{l\in\bar{\mathbb{Z}}}\left(\sum_{|j_1|\ge|j_2|\ge|j_3|}\frac{|l|^{s'}\mu(j,l)^\nu}{C(j,l)^\beta}A(j,l)^N|z_{j_1}||z_{j_2}||z_{j_3}|\right)^2, \qquad (2.15)
\end{aligned}$$

where $\bar{\mathbb{Z}}^k_>$ denotes the set of *ordered* k-uples $(j_1,\dots,j_k)$ such that $|j_1|\ge|j_2|\ge\dots\ge|j_k|$. We used the following result in the last inequality:

Lemma 2.15. — *Given any $s\ge 0$, $s_0>\frac{1}{2}$ and $z\in\ell^2_{s+s_0}$ we have*

$$\sum_{j\in\bar{\mathbb{Z}}}|j|^s|z_j|\le C_{s_0}||z||_{s+s_0}\,.$$

Proof. — This result is a simple consequence of Cauchy-Schwarz inequality :

$$\sum_{j\in\bar{\mathbb{Z}}}|j|^s|z_j|=\sum_{j\in\bar{\mathbb{Z}}}\frac{1}{|j|^{s_0}}|j|^{s+s_0}|z_j|\le\left(\sum_{j\in\bar{\mathbb{Z}}}\frac{1}{|j|^{2s_0}}\right)^{\frac{1}{2}}||z||_{s+s_0}\,.$$

□

Before continuing with the proof of assertion (ii) of Proposition 2.13, we give two technical lemmas which give an estimate of $A(j,l)$.

Lemma 2.16. — *Given any ordered k-tuple $j\in\bar{\mathbb{Z}}^k_>$ and $l\in\bar{\mathbb{Z}}$, we have*

$$|l|A(j,l)\le 2|j_1|\,.$$

Proof. — It is straightforward if $|l|\le 2|j_1|$, since $A(j,l)\le 1$. If not, the order is the following : $|l|>2|j_1|>|j_1|\ge|j_2|$ and

$$|l|A(j,l)=\frac{|l|\sqrt{|j_1j_2|}}{\sqrt{|j_1j_2|}+|l|-|j_1|}\le\frac{|l|\sqrt{|j_1j_2|}}{|l|/2}\le 2|j_1|\,,$$

and the lemma is proved. □

Lemma 2.17. — *Given any ordered k-uple $j \in \bar{\mathbb{Z}}^k_>$ and $l \in \bar{\mathbb{Z}}$ we have*

$$A(j,l) \leq \tilde{A}(j_1,j_2,l) := \begin{cases} 2\frac{|j_2|}{|l|+|j_1|-|j_2|} & \text{if } |l| \leq |j_2|\,, \\ 2\frac{\sqrt{|lj_2|}}{\sqrt{|lj_2|}+||j_1|-|l||} & \text{if } |l| \geq |j_2|\,. \end{cases}$$

Proof. — If $|l| > 2|j_1|$, $A(j,l)$ reads :

$$A(j,l) = \frac{\sqrt{|j_1 j_2|}}{\sqrt{|j_1 j_2|} + |l| - |j_1|}\,.$$

We can write :

$$\begin{aligned} \sqrt{|j_1 j_2|} + |l| - |j_1| &= \sqrt{|lj_2|} + |l| - |j_1| - \sqrt{|j_2|}(\sqrt{|l|} - \sqrt{|j_1|}) \\ &= \sqrt{|lj_2|} + |l| - |j_1| - \sqrt{|j_2|}\frac{|l| - |j_1|}{\sqrt{|l|} + \sqrt{|j_1|}} \\ &\geq \sqrt{|lj_2|} + |l| - |j_1| - \sqrt{|j_1|}\frac{|l| - |j_1|}{\sqrt{|l|} + \sqrt{|j_1|}} \\ &\geq \sqrt{|lj_2|} + \frac{\sqrt{2}}{\sqrt{2}+1}(|l| - |j_1|) \end{aligned}$$

Hence,

$$A(j,l) \leq \frac{1+\sqrt{2}}{\sqrt{2}} \frac{\sqrt{|j_1 j_2|}}{\sqrt{|j_1 j_2|} + |l| - |j_1|} \leq 2\frac{\sqrt{|lj_2|}}{\sqrt{|lj_2|} + |l| - |j_1|}\,.$$

If $|j_2| \leq |l| \leq 2|j_1|$, then $B(j,l)^2 = |j_2| \min(|l|, |j_1|) \in \left[\frac{|lj_2|}{2}, |lj_2|\right]$, therefore

$$A(j,l) \leq \frac{\sqrt{|lj_2|}}{1/\sqrt{2}\sqrt{|lj_2|} + ||l| - |j_1||} \leq 2\frac{\sqrt{|lj_2|}}{\sqrt{|lj_2|} + ||l| - |j_1||}\,.$$

Finally, if $|l| \leq |j_2|$ we get

$$A(j,l) = \frac{\sqrt{|lj_2|}}{\sqrt{|lj_2|} + |j_1| - |j_2|} \leq 2\frac{|j_2|}{|l| + |j_1| - |j_2|}\,,$$

and this ends the proof of Lemma 2.17. □

To continue with the proof of assertion (ii) of Proposition 2.13, we define $0 < \varepsilon < s - s' - \frac{1}{2}$, and $N = s + 1 + \varepsilon$. In view of (2.15), we may decompose :

$$||X_P(z)||^2_{s'} \leq C \sum_{l \in \bar{\mathbb{Z}}} (T_1(l) + T_2(l))^2\,, \tag{2.16}$$

with

$$T_1(l) = \sum_{|j_1|\geq|j_2|\geq|j_3|,|j_2|>|l|} \frac{|l|^{s'}|\mu(j,l)|^\nu}{\max(|j_1|,|l|)^\beta} A(j,l)^N |z_{j_1}||z_{j_2}||z_{j_3}|$$

$$T_2(l) = \sum_{|j_1|\geq|j_2|\geq|j_3|,|j_2|\leq|l|} \frac{|l|^{s'}|\mu(j,l)|^\nu}{\max(|j_1|,|l|)^\beta} A(j,l)^N |z_{j_1}||z_{j_2}||z_{j_3}|\,.$$

Since $A(j,l) \leq 1$ and $N > \frac{1}{2} + s' + \varepsilon$, we may estimate $T_1(l)$ using Lemmas 2.16 and 2.17 :

$$\begin{aligned} T_1(l) &\leq C \sum_{|j_1|\geq|j_2|\geq|j_3|,|j_2|>|l|} |j_1|^{s'}|j_2|^\nu \tilde{A}(j_1,j_2,l)^{\frac{1}{2}+\varepsilon}|z_{j_1}||z_{j_2}||z_{j_3}| \\ &\leq C||z||_{s_0} \sum_{|j_1|\geq|j_2|,|j_2|>|l|} \frac{1}{|l|^{\frac{1}{2}+\varepsilon}} |j_1|^{s-\frac{1}{2}-\varepsilon}|z_{j_1}||j_2|^{\nu+\frac{1}{2}+\varepsilon}|z_{j_2}| \\ &\leq C||z||_{s_0} \frac{1}{|l|^{\frac{1}{2}+\varepsilon}} ||z||_s ||z||_{\nu+1+2\varepsilon}\,, \end{aligned}$$

hence $T_1(l)$ is an ℓ^2-sequence, whose ℓ^2-norm is bounded above by $C||z||_{s_0}^2||z||_s$ if we assume that $s_0 > \nu + 1 + 2\varepsilon$. Concerning $T_2(l)$, using Lemmas 2.16 and 2.17, we obtain

$$\begin{aligned} T_2(l) &\leq C \sum_{|j_1|\geq|j_2|\geq|j_3|,|j_2|\leq|l|} \frac{1}{|l|^{s-s'+\beta}} |j_1|^s|j_2|^\nu \tilde{A}(j_1,j_2,l)^{N-s}|z_{j_1}||z_{j_2}||z_{j_3}| \\ &\leq C\frac{||z||_{s_0}}{|l|^{s-s'+\beta}} \sum_{|j_1|\geq|j_2|,|j_2|\leq|l|} \left(\frac{\sqrt{|lj_2|}}{1+||j_1|-|l||}\right)^{1+\varepsilon} |j_1|^s|z_{j_1}||j_2|^\nu|z_{j_2}| \\ &\leq C\frac{||z||_{s_0}}{|l|^{s-s'+\beta-(1+\varepsilon)/2}} \left(\sum_{j_2\in\mathbb{Z}} |j_2|^{\nu+(1+\varepsilon)/2}|z_{j_2}|\right) \sum_{j_1\in\mathbb{Z}} \frac{|j_1|^s|z_{j_1}|}{(1+||j_1|-|l||)^{1+\varepsilon}}. \end{aligned}$$

The last sum in j_1 is a convolution product of the ℓ^2-sequence $|j_1|^s|z_{j_1}|$ and the ℓ^1-sequence $\frac{1}{(1+|j_1|)^{1+\varepsilon}}$ and thus a ℓ^2-sequence with respect to the index l, whose ℓ^2-norm is bounded by $||z||_s$. Choosing $\varepsilon > 0$ in such a way that $s - s' + \beta - (1+\varepsilon)/2 > 0$, the sequence $T_2(l)$ is in ℓ^2, with a norm bounded by

$$||T_2|| \leq C||z||_{s_0}||z||_{\nu+(1+\varepsilon)/2}||z||_s \leq C||z||_{s_0}^2||z||_s\,,$$

with $s_0 > \nu + (1+\varepsilon)/2$. Collecting the estimates for T_1 and T_2, we obtain the desired inequality.

(iii) We define $0 < \varepsilon < 1/12$ and $N = s + \frac{1}{2} + \varepsilon$. We have, as in (ii), this first estimate
(2.17)

$$||X_P(z)||_s^2 \leq C||z||_{s_0}^{2(k-3)} \sum_{l\in\bar{\mathbb{Z}}} \left(\sum_{|j_1|\geq|j_2|\geq|j_3|} \frac{|l|^s \mu(j,l)^\nu}{C(j,l)^\beta(1+S(j,l))} A(j,l)^N |z_{j_1}||z_{j_2}||z_{j_3}| \right)^2 .$$

As in (ii), we may also decompose the sum on j_1, j_2 and j_3 into two pieces, $T_1^+(l)$ collecting all the terms with $|j_2| > |l|$ and $T_2^+(l)$ collecting those with $|j_2| \leq |l|$. Following (ii), since $C(j,l) \geq 1$ and $1 + S(j,l) \geq 1$, we obtain for T_1^+ :

$$\begin{aligned}
T_1^+(l) &\leq C \sum_{|j_1|\geq|j_2|\geq|j_3|,|j_2|>|l|} |l|^s |j_2|^\nu A(j,l)^N |z_{j_1}||z_{j_2}||z_{j_3}| \\
&\leq C \sum_{|j_1|\geq|j_2|\geq|j_3|,|j_2|>|l|} |l|^{1/2+\varepsilon} |j_1|^{s-1/2-\varepsilon} |j_2|^\nu A(j,l)^{N-(s-1/2-\varepsilon)} |z_{j_1}||z_{j_2}||z_{j_3}| \\
&\leq C||z||_{s_0} \sum_{|j_1|\geq|j_2|,|j_2|>|l|} |l|^{1/2+\varepsilon} |j_1|^{s-1/2-\varepsilon} |j_2|^\nu \tilde{A}(j_1,j_2,l)^{N-(s-1/2-\varepsilon)} |z_{j_1}||z_{j_2}| \\
&\leq C||z||_{s_0} \sum_{|j_1|\geq|j_2|,|j_2|>|l|} |l|^{s-N} |j_1|^{s-1/2-\varepsilon} |j_2|^{\nu+N-(s-1/2-\varepsilon)} |z_{j_1}||z_{j_2}| \\
&\leq C||z||_{s_0} \frac{1}{|l|^{\frac{1}{2}+\varepsilon}} \sum_{|j_1|\geq|j_2|,|j_2|>|l|} |j_1|^{s-1/2-\varepsilon} |j_2|^{\nu+N-(s-1/2-\varepsilon)} |z_{j_1}||z_{j_2}| \\
&\leq C||z||_{s_0} \frac{1}{|l|^{\frac{1}{2}+\varepsilon}} ||z||_s ||z||_{\nu+1+2\varepsilon} ,
\end{aligned}$$

hence $T_1^+(l)$ is a ℓ^2-sequence, whose ℓ^2-norm is bounded above by $C||z||_{s_0}^2||z||_s$ if $s_0 > \nu + 1 + 2\varepsilon$.

The estimate on T_2^+ will need all factors assigned in the definition of $\mathcal{T}^{\nu,\beta,+}$:

$$\begin{aligned}
T_2^+(l) &\leq C \sum_{|j_1|\geq|j_2|\geq|j_3|,|j_2|\leq|l|} \frac{|j_1|^s |j_2|^\nu}{\max(j_1,l)^\beta(1+||j_1|-|l||)} \tilde{A}(j_1,j_2,l)^{N-s} |z_{j_1}||z_{j_2}||z_{j_3}| \\
&\leq C||z||_{s_0} \sum_{|j_1|\geq|j_2|,|j_2|\leq|l|} \left(\frac{\sqrt{|lj_2|}}{1+||j_1|-|l||} \right)^\varepsilon \frac{1}{|l|^\beta(1+||j_1|-|l||)} |j_1|^s |z_{j_1}||j_2|^\nu |z_{j_2}| \\
&\leq C||z||_{s_0} \frac{1}{|l|^{\beta-\varepsilon/2}} \sum_{j_2\in\bar{\mathbb{Z}}} |j_2|^{\nu+\varepsilon/2} |z_{j_2}| \sum_{j_1\in\bar{\mathbb{Z}}} \frac{|j_1|^s |z_{j_1}|}{(1+||j_1|-|l||)^{1+\varepsilon}} .
\end{aligned}$$

Once again, the last sum in j_1 is a convolution product of the ℓ^2 sequence $|j_1|^s|z_{j_1}|$ and the ℓ^1 sequence $\frac{1}{(1+|j_1|)^{1+\varepsilon}}$. Choosing $\varepsilon > 0$ in such a way that

$\beta-\varepsilon/2>0$, the sequence $T_2^+(l)$ is in ℓ^2, with a norm bounded by

$$||T_2||\leq C||z||_{s_0}||z||_{\nu+(1+\varepsilon)/2}||z||_s\leq C||z||_{s_0}^2||z||_s\,,$$

with $s_0>\nu+(1+\varepsilon)/2$. Collecting the estimates for T_1^+ and T_2^+, we obtain the stated inequality.

(iv) Let $k+1=2m$. As in (ii), we obtain
(2.18)

$$||X_P||_s^2\leq C\sum_{l\in\bar{\mathbb{Z}}}\left(\sum_{j\in\bar{\mathbb{N}}_>^{m-1}}|l|^s|z_l|\frac{\mu(j,j,l,l)^\nu}{C(j,j,l,l)^\beta}|z_{j_1}||z_{-j_1}|\ldots|z_{j_{m-1}}||z_{-j_{m-1}}|\right)^2,$$

using the same convention for $\mu(j,j,l,l)$ and $C(j,j,l,l)$ as for $\mu(j,l)$ and $C(j,l)$: as an example, $\mu(j,j,l,l)$ is the third biggest integer among $|j_1|,|j_1|,\ldots|j_{m-1}|,|j_{m-1}|,|l|$ and $|l|$, that is, if j is ordered, either $\mu(j,j,l,l)=|j_1|$, and in this case $C(j,j,l,l)=|l|$, or $\mu(j,j,l,l)=|l|$ and in this case $C(j,j,l,l)=|j_1|$. Notice that $A(j,j,l,l)=1$ does not help for this computation. The sum over j can be decomposed into two parts :

$$\begin{aligned}
&\sum_{j\in\bar{\mathbb{N}}_>^{m-1},j_1\leq l}|l|^s|z_l|\frac{\mu(j,j,l,l)^\nu}{C(j,j,l,l)^\beta}|z_{j_1}||z_{-j_1}|\ldots|z_{j_{m-1}}||z_{-j_{m-1}}|\\
&\leq\sum_{j\in\bar{\mathbb{N}}_>^{m-1},j_1\leq l}|l|^s|z_l|\frac{|j_1|^\nu}{|l|^\beta}|z_{j_1}||z_{-j_1}|\ldots|z_{j_{m-1}}||z_{-j_{m-1}}|\\
&\leq|l|^{s-\beta}|z_l|\sum_{j_1}j_1^\nu|z_{j_1}||z_{-j_1}|||z||_0^{2(m-2)}\\
&\leq|l|^{s-\beta}|z_l|||z||_{\nu/2}^2||z||_0^{2(m-2)},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j\in\bar{\mathbb{N}}_>^{m-1},j_1>l}|l|^s|z_l|\frac{\mu(j,j,l,l)^\nu}{C(j,j,l,l)^\beta}|z_{j_1}||z_{-j_1}|\ldots|z_{j_{m-1}}||z_{-j_{m-1}}|\\
&\leq|l|^s|z_l|\sum_{j\in\bar{\mathbb{N}}_>^{m-1}}|j_1|^{\nu-\beta}|z_{j_1}||z_{-j_1}|\ldots|z_{j_{m-1}}||z_{-j_{m-1}}|\\
&\leq|l|^s|z_l|||z||_{(\nu-\beta)/2}^2||z||_0^{2(m-2)}
\end{aligned}$$

Inserting these two estimates in (2.18) we get (2.12). □

The second essential property satisfied by polynomials in $\mathcal{T}_k^{\nu,\beta}$ is captured in the following

Proposition 2.18. — *Let* $k_1, k_2 \geq 2$, $\nu_1, \nu_2 \geq 0$ *and* $\beta > 0$ *The map* $(P,Q) \mapsto \{P,Q\}$ *defines a continuous map from* $\mathcal{T}_{k_1+1}^{\nu_1,\beta,+} \times \mathcal{T}_{k_2+1}^{\nu_2,\beta}$ *to* $\mathcal{T}_{k_1+k_2}^{\nu',\beta}$ *for* $\nu' = 2(\nu_1+\nu_2)+1$.

Proof. — We assume that $P \in \mathcal{T}_{k_1+1}^{\nu_1,\beta,+}$ and $Q \in \mathcal{T}_{k_2+1}^{\nu_2,\beta}$ are homogeneous polynomials and we write

$$P(z) = \sum_{j \in \bar{\mathbb{Z}}^{k_1+1}} a_j \, z_{j_1} \dots z_{j_{k_1+1}}$$

and

$$Q(z) = \sum_{i \in \bar{\mathbb{Z}}^{k_2+1}} b_i \, z_{i_1} \dots z_{i_{k_2+1}} \ .$$

In view of the symmetry of the estimate (2.7) with respect to the involved indices, one easily obtains

$$\{P,Q\}(z) = \sum_{(j,i) \in \bar{\mathbb{Z}}^{k_1+k_2}} c_{j,i} \, z_{j_1} \dots z_{j_{k_1}} z_{i_1} \dots z_{i_{k_2}}$$

with

$$|c_{j,i}| \leq c_{N,N'} \sum_{l \in \bar{\mathbb{Z}}} \frac{\mu(j,l)^{\nu_1}}{C(j,l)^\beta (1+S(j,l))} A(j,l)^N \frac{\mu(i,l)^{\nu_2}}{C(i,l)^\beta} A(i,l)^{N'}.$$

Therefore it remains to prove that, for each $M \geq 1$, there exist $N, N' \geq 1$, $C > 0$ such that for all $j \in \bar{\mathbb{Z}}^{k_1}$ and all $i \in \bar{\mathbb{Z}}^{k_2}$,

(2.19)
$$\sum_{l \in \bar{\mathbb{Z}}} \frac{\mu(j,l)^{\nu_1}}{C(j,l)^\beta (1+S(j,l))} A(j,l)^N \frac{\mu(i,l)^{\nu_2}}{C(i,l)^\beta} A(i,l)^{N'} \leq C \frac{\mu(j,i)^{\nu'}}{C(i,j)^\beta} A(j,i)^M$$

with $\nu' = 2(\nu_1+\nu_2)+1$.

In order to simplify the notations, and because it does not change the estimates of (2.19), we will assume that $k_1 = k_2 = k$. We can also assume by symmetry that

- all the indices are positive: $j_1, \dots, j_k, i_1, \dots, i_k \geq 1$.
- j and i are ordered: $j_1 \geq \dots \geq j_k$ and $i_1 \geq \dots \geq i_k$.

We begin with two technical lemmas whose proofs are postponed at the end of this proof.

Lemma 2.19. — *There is a constant* $C > 0$ *such that for any* $j \in \bar{\mathbb{Z}}^{k_1}$, $i \in \bar{\mathbb{Z}}^{k_2}$ *and* $l \in \bar{\mathbb{Z}}$ *we have*

$$A(j,l)^2 A(i,l)^2 \leq C A(i,j). \tag{2.20}$$

Lemma 2.20. — *There is a constant $C > 0$ such that for any $j \in \bar{\mathbb{Z}}^{k_1}$, $i \in \bar{\mathbb{Z}}^{k_2}$ and $l \in \bar{\mathbb{Z}}$ we have*

$$\max(\mu(j,l)A(i,l)^2, \mu(i,l)A(j,l)^2) \leq C\mu(i,j)^2. \tag{2.21}$$

Using these lemmas, in order to prove (2.19), it suffices to prove

$$\sum_{l\in\bar{\mathbb{Z}}} \frac{1}{C(j,l)^\beta(1+S(j,l))} \frac{A(i,l)^2}{C(i,l)^\beta} \leq C\frac{\mu(j,i)}{C(i,j)^\beta}.$$

Noticing that $C(i,l)C(j,l) \geq C(i,j)l$, it suffices to verify that

$$\sum_{l\in\bar{\mathbb{Z}}} \frac{A(i,l)^2}{(1+S(j,l))l^\beta} \leq C\mu(j,i).$$

Decompose the sum in two parts, $I_1 = \sum_{l>j_2}$ and $I_2 = \sum_{l\leq j_2}$. For the first sum we have

$$I_1 = \sum_{l>j_2} \frac{A(i,l)^2}{(1+S(j,l))l^\beta} \leq \sum_{l\in\bar{\mathbb{Z}}} \frac{1}{(1+|l-j_1|)l^\beta} \leq C,$$

while for the second one

$$I_2 = \sum_{l\leq j_2} \frac{A(i,l)^2}{(1+S(j,l))l^\beta} \leq \sum_{l\leq j_2} \frac{A(i,l)^2}{l^\beta}.$$

In this last sum, if $j_2 < \mu(i,j)$, then

$$I_2 \leq j_2 \leq \mu(i,j).$$

On the other hand, if $\mu(i,j) \leq j_2$, then we decompose the I_2 sum in two parts, $I_{2,1} = \sum_{l<2i_1}$ and $I_{2,2} = \sum_{l\geq 2i_1}$. Since $i_1 \leq \mu(i,j) = \max(i_1, j_3)$ we have

$$I_{2,1} = \sum_{l\leq 2i_1} \frac{A(i,l)^2}{l^\beta} \leq 2i_1 \leq 2\mu(i,j).$$

Finally, when $l \geq 2i_1$ we have $S(i,l) \geq l/2$ and $B(i,l)^2 = i_1 i_2 \leq i_2 l/2 \leq \mu(i,j)l/2$ and thus $A(i,l) \leq \sqrt{2\mu(i,j)}l^{-1/2}$ which leads to

$$I_{2,2} = \sum_{2i_1\leq l\leq j_2} \frac{A(i,l)^2}{l^\beta} \leq C\mu(i,j)\sum_{l\in\bar{\mathbb{N}}} \frac{1}{l^{1+\beta}} \leq C\mu(i,j).$$

□

Proof of lemma 2.19 — The estimate (2.20), being symmetric with respect to i and j, we can assume that $j_1 \geq i_1$. We consider three cases, depending of the position of l with respect to i_1 and j_1.

First case $l \geq j_1$:

We have $S(i,l) = |i_1 - l| \geq |i_1 - j_1| \geq S(i,j)$ and $B(i,l) = (i_1 i_2)^{1/2} \leq B(i,j)$. Therefore

$$A(i,l) = \frac{B(i,l)}{B(i,l)+S(i,l)} \leq \frac{B(i,l)}{B(i,l)+S(i,j)} \leq \frac{B(i,j)}{B(i,j)+S(i,j)} = A(i,j),$$

and using $A(j,l) \leq 1$, (2.20) is proved.

Second case $l \leq i_1$:
Similarly as in the first case, we have $S(j,l) \geq S(i,j)$ and $B(j,l) = (j_2 \max(j_3,l))^{1/2} \leq (j_2 \max(j_3,i_1))^{1/2} \leq B(i,j)$ and thus

$$A(j,l) \leq A(i,j).$$

Third case $i_1 < l < j_1$:
That is the most complicated case and we have to distinguish whether $i_1 \geq j_2$ or not.
Subcase 1. $i_1 \geq j_2$:
We have $B(i,l) \leq B(i,j)$ thus if $S(i,l) = |i_1 - l| \geq \frac{1}{2}|i_1 - j_1| = \frac{1}{2}S(i,j)$ we obtain $A(j,l) \leq 2A(i,j)$ and (2.20) holds true. Now if $S(i,l) < \frac{1}{2}S(i,j)$ then $S(j,l) \geq \frac{1}{2}S(i,j)$ since $S(i,l)+S(j,l) \geq S(i,j)$. Furthermore, if $B(j,l) \leq B(i,j)$ then

$$A(j,l) = \frac{B(j,l)}{B(j,l)+S(j,l)} \leq 2\frac{B(j,l)}{B(j,l)+S(i,j)} \leq 2\frac{B(i,j)}{B(i,j)+S(i,j)} = 2A(i,j)\,,$$

and (2.20) holds. If $B(j,l) > B(i,j)$, then using

$$B(j,l)^2 = j_2 l = j_2 i_1 + j_2(l-i_1) \leq B(i,j)^2 + j_2 S(i,l) \leq B(i,j)^2 + \frac{1}{2}B(i,j)S(i,j)\,,$$

we deduce

$$A(j,l)^2 \leq \frac{B(j,l)^2}{(B(i,j)+\frac{1}{2}S(i,j))^2} \leq 2\frac{B(i,j)^2 + B(i,j)S(i,j)}{(B(i,j)+S(i,j))^2} \leq 2(A(i,j)^2 + A(i,j))\,,$$

thus (2.20) is also satisfied in this case, since $A(i,j) \leq 1$.

Subcase 2. $i_1 \leq j_2$:
We still have $B(i,l) \leq B(i,j)$ thus if furthermore $S(i,j) \leq 2S(i,l)$ then $A(i,l) \leq 2A(i,j)$ and (2.20) is true. So we assume $2S(i,l) < S(i,j)$ which implies $S(i,j) \leq 2S(j,l)$ since $S(i,l)+S(j,l) \geq S(i,j)$. If furthermore $l \leq j_3$, $B(j,l) = B(j) \leq B(i,j)$ and thus $A(j,l) \leq 2A(i,j)$ and (2.20) is again true. So we assume $j_3 \leq l$ and we have

$$B(j,l)^2 = l j_2 = i_1 j_2 + j_2(l-i_1) \leq B(i,j)^2 + j_2 S(i,l).$$

If $S(i,l) \leq l/2$ then we deduce $B(j,l)^2 \leq 2B(j,l)^2$ and (2.20) is satisfied. It remains to consider the case $S(i,l) > l/2$ which implies $i_1 < l/2$ and thus

$$A(i,l) \leq \frac{i_1}{i_1 + l/2} \leq 2\frac{i_1}{l}. \tag{2.22}$$

Let $n \geq 1$ such that $\frac{l}{2^{n+1}} \leq i_1 \leq \frac{l}{2^n}$ we get from (2.22)

$$A(i,l) \leq \frac{1}{2^{n-1}}. \tag{2.23}$$

On the other hand

$$A(j,l) \leq 2\frac{(lj_2)^{1/2}}{(lj_2)^{1/2} + S(i,j)} \leq 2\frac{(lj_2)^{1/2}}{(i_1 j_2)^{1/2} + S(i,j)} \tag{2.24}$$

and

$$A(i,j) \geq \frac{(i_1 j_2)^{1/2}}{(i_1 j_2)^{1/2} + S(i,j)} \geq \frac{1}{2^{n+1}} \frac{(lj_2)^{1/2}}{(i_1 j_2)^{1/2} + S(i,j)}. \tag{2.25}$$

Combining (2.23), (2.24) and (2.25) we conclude

$$A(i,l)A(j,l) \leq 8A(i,j).$$

Proof of lemma 2.20 — The estimate (2.21) being symmetric with respect to i and j, we can assume $j_1 \geq i_1$. If furthermore $i_1 \geq j_2$ then one easily verifies that

$$\mu(i,l) \leq \mu(i,j) \text{ and } \mu(j,l) \leq \mu(i,j)$$

and estimates (2.21) is satisfied.
In the case $j_1 \geq j_2 \geq i_1$ we still have $\mu(i,l) \leq \mu(i,j)$ but $\mu(j,l)$ could be larger than $\mu(i,j)$. Actually if $\mu(j,l) \leq 2\mu(i,j)$, estimates (2.21) is still trivially satisfied. Therefore it remains to consider the case where $\mu(j,l) > 2\mu(i,j)$. Remark that in this case $i_1 \leq \mu(i,j) \leq \frac{\mu(j,l)}{2} \leq l/2$ and thus $S(i,l) = |i_1 - l| \geq l/2$ which leads to

$$A(i,l) \leq \frac{(i_1 i_2)^{1/2}}{S(i,l)} \leq \frac{(2i_2)^{1/2}}{l^{1/2}} \leq \frac{(2\mu(i,j))^{1/2}}{l^{1/2}}.$$

Using this last estimates one gets

$$\mu(j,l)A(i,l)^2 \leq lA(i,l)^2 \leq 2\mu(i,j)^2.$$

We end this section with a proposition concerning Lie transforms of homogeneous polynomials $\chi \in \mathcal{T}_l^{\delta,\beta,+}$, i.e. time 1 flow of the Hamiltonian vector field X_χ.

Proposition 2.21. — *Let χ be a real homogeneous polynomial in $\mathcal{T}_l^{\delta,\beta,+}$ with $\delta \geq 0$, $\beta > 0$, $l \geq 3$ take $s > s_1 := \delta + 3/2$ and denote by ϕ the Lie transform associated with χ. We have*

(i) *ϕ is an analytic canonical transformation from an open ball B_ϵ of center 0 and radius ϵ in $\mathcal{P}_s$ into the open ball $B_{2\epsilon}$ in $\mathcal{P}_s$ satisfying*

$$\|\phi(z) - z\|_s \leq C_s \|z\|_s^2 \text{ for any } z \in B_\varepsilon. \tag{2.26}$$

In particular if $F \in \mathcal{H}^s$ with $s > s_1$ then $F \circ \phi \in \mathcal{H}^s$. Furthermore, if F is real then $F \circ \phi$ is real too.

(ii) *Let $P \in \mathcal{T}_n^{\nu,\beta} \cap \mathcal{H}^s$, $\nu \geq 0$, $n \geq 3$ and fix $r \geq n$ an integer. Then*

$$P \circ \phi = Q_r + R_r$$

where:
- Q_r is a polynomial of degree at most r, belonging to $\mathcal{T}^{\nu',\beta} \cap \mathcal{H}^s$ with $\nu' = 2^{r-n}\nu + (2^{r-n} - 1)(2\beta + 1)$,
- R_r is a Hamiltonian in $\mathcal{T}^{\nu'',\beta} \cap \mathcal{H}^s$ with $\nu'' = 2^{r-n+1}\nu + (2^{r-n+1} - 1)(2\beta + 1)$, having a zero of order $r + 1$ at the origin.

Proof. — (i) Since $\chi \in \mathcal{T}_l^{\delta,\beta,+}$, by Proposition 2.13(iii), $X_\chi \in C^\infty(\mathcal{P}_s, \mathcal{P}_s)$ for $s > s_1 = \delta + 3/2$. In particular, for $s > s_1$, the flow Φ^t generated by the vector field X_χ transports an open neighborhood of the origin in $\mathcal{P}_s$ into an open neighborhood of the origin in $\mathcal{P}_s$. Notice that since χ is real, Φ^t transports the "real part" of $\mathcal{P}_s$, $\{(\xi, \bar{\xi}) \in \mathcal{P}_s\}$, into itself. Furthermore one has for $z \in \mathcal{P}_s$ small enough

$$\Phi^t(z) - z = \int_0^t X_\chi(\Phi^{t'}(z))dt'$$

and since χ has a zero of order 3 at least, one gets by Proposition 2.13(iii),

$$\left\|\Phi^t(z) - z\right\|_s \leq C_s \int_0^t \left\|\Phi_\chi^{t'}(z)\right\|_s^2 dt'.$$

Then, by a classical continuity argument, there exists $\epsilon > 0$ such that the flow $B_\varepsilon \ni z \mapsto \Phi_\chi^t(z) \in B_{2\varepsilon}$ is well defined and smooth for $0 \leq t \leq 1$. Furthermore, the Lie transform $\phi = \Phi^1$ satisfies (2.26).
On the other hand, by simple composition we get that if $F \in \mathcal{H}^s$ with $s > s_1$, then $F \circ \phi \in C^\infty(B_\epsilon, \mathbb{C})$. In view of the formula

$$X_{F\circ\phi}(z) = (D\phi(z))^{-1} X_F(\phi(z)),$$

we deduce that $X_{F\circ\phi} \in C^\infty(B_\epsilon, \mathcal{P}_s)$. We now have to check the properties concerning the Taylor polynomials of $F \circ \phi$. Denoting by F_k (resp. $(F \circ \phi)_k$)

the homogeneous polynomial of degree k appearing in the Taylor expansion of F (resp $F\circ\phi$), and putting $F_k^{[0]}=F_k$, $F_k^{[j+1]}=\{F_k,\chi\}$, we have

$$(F\circ\phi)_k(z)=\sum_{j\geq 0,k'\geq 0,k'+j(l-2)=k}F_{k'}^{[j]}(z),$$

since χ is itself a homogeneous polynomial of degree l. It is then sufficient to prove that the Poisson bracket of a homogeneous polynomial F_k in $\mathcal{H}^s$ with χ stays in $\mathcal{H}^s$.
Using the (constant) symplectic form ω on $\mathcal{P}_s$, we get $\{F_k,\chi\}(z)=\omega(X_{F_k},X_\chi)$, and so $\{F_k,\chi\}\in C^\infty(B_\epsilon,\mathbb{C})$. Moreover

$$X_{\{F_k,\chi\}}(z)=[X_{F_k},X_\chi]=\lim_{t\to 0}\frac{1}{t}(X_{F_k}-\Phi^t_*(X_{F_k}))(z)\,.$$

Since Φ^t is the flow of the regular Hamiltonian $\chi\in C^\infty(B_\epsilon,\mathbb{R})$, the Cauchy Lipschitz theorem implies that the mapping $(t,z)\mapsto\Phi^t_*(X_{F_k})(z)$ is in $C^\infty([-1,1]\times B_\epsilon,\mathcal{P}_s)$. Now, $X_{\{F_k,\chi\}}$ is nothing else but the time derivative of this mapping at time 0, hence $X_{\{F_k,\chi\}}\in C^\infty(B_\epsilon,\mathcal{P}_s)$ and the claim is proved.

(ii) By a direct calculus one has

$$\frac{d^k}{dt^k}P\circ\Phi^t(z)\big|_{t=0}=P^{[k]}(z)$$

with the same notation $P^{[k+1]}=\{P^{[k]},\chi\}$ and $P^{[0]}=P$. Therefore applying the Taylor's formula to $P\circ\Phi^t(z)$ between $t=0$ and $t=1$ we deduce

$$P\circ\phi(z)=\sum_{k=0}^{r-n}\frac{1}{n!}P^{[k]}(z)+\frac{1}{(r-n)!}\int_0^1(1-t)^rP^{[r-n+1]}(\Phi^t(z))dt. \tag{2.27}$$

Notice that $P^{[k]}(z)$ is a homogeneous polynomial of degree $n+k(l-2)$ and, by Proposition 2.18, $P^{[k]}(z)\in\mathcal{T}^{2^k\nu+(2^k-1)(2\delta+1),\beta}$. Moreover $P^{[k]}(z)$ is a homogeneous polynomial in the Taylor expansion of $P\circ\phi\in\mathcal{H}^s$, hence it is in $\mathcal{H}^s$. Therefore (2.27) decomposes in the sum of a polynomial of degree r in $\mathcal{T}_r^{\nu',\beta}$, and a function in $\mathcal{H}^s$ having a zero of degree $r+1$ at the origin. ☐

2.3. The Birkhoff normal form theorem. — We start with the resolution of the homological equation and then state the normal form theorem.

Lemma 2.22. — *Let $\nu\in[0,+\infty)$ and assume that the frequency vector of H_0 is strongly non resonant (see Definition 2.5). Let Q be a homogeneous real polynomial of degree k in $\mathcal{T}_k^{\nu,\beta}$, there exist $\nu'>\nu$, and Z and χ two* *homogeneous real polynomials of degree k, respectively in $\mathcal{T}_k^{\nu',\beta}$ and $\mathcal{T}_k^{\nu',\beta,+}$, which satisfy*

$$\{H_0, \chi\} + Q = Z \tag{2.28}$$

and

$$\{Z, I_j\} = 0 \quad \forall j \geq 1 \tag{2.29}$$

and thus Z is in normal form. Furthermore, Z and χ both belong to $\mathcal{H}^s$ for $s > \nu' + 1$.

Proof. — For $j \in \bar{\mathbb{N}}^{k_1}$ and $l \in \bar{\mathbb{N}}^{k_2}$ with $k_1 + k_2 = k$ we denote

$$\xi^{(j)}\eta^{(l)} = \xi_{j_1} \ldots \xi_{j_{k_1}} \eta_{l_1} \ldots \eta_{l_{k_2}}.$$

One has

$$\{H_0, \xi^{(j)}\eta^{(l)}\} = -i\Omega(j,l)\xi^{(j)}\eta^{(l)}$$

with

$$\Omega(j,l) := \omega_{j_1} + \ldots + \omega_{j_{k_1}} - \omega_{l_1} - \ldots - \omega_{l_{k_2}}.$$

Let $Q \in \mathcal{T}_k^{\nu,\beta}$

$$Q = \sum_{(j,l) \in \bar{\mathbb{N}}^k} a_{jl} \xi^{(j)}\eta^{(l)}$$

where $(j,l) \in \bar{\mathbb{N}}^k$ means that $j \in \bar{\mathbb{N}}^{k_1}$ and $l \in \bar{\mathbb{N}}^{k_2}$ with $k_1 + k_2 = k$. Let us define

$$b_{jl} = i\Omega(j,l)^{-1} a_{jl}, \quad c_{jl} = 0 \quad \text{when } \{j_1, \ldots, j_{k_1}\} \neq \{l_1, \ldots, l_{k_2}\} \tag{2.30}$$

and

$$c_{jl} = a_{jl}, \quad b_{jl} = 0 \quad \text{when } \{j_1, \ldots, j_{k_1}\} = \{l_1, \ldots, l_{k_2}\}. \tag{2.31}$$

As ω is strongly non resonant, there exist γ and α such that

$$|\Omega(j,l)| \geq \gamma \frac{1 + S(j,l)}{\mu(j,l)^\alpha}$$

for all $(j,l) \in \bar{\mathbb{N}}^k$ with $\{j_1, \ldots, j_{k_1}\} \neq \{l_1, \ldots, l_{k_2}\}$. Thus, in view of Definitions 2.6 and 2.7, the polynomial

$$\chi = \sum_{(j,l) \in \bar{\mathbb{N}}^k} b_{j,l} \xi^{(j)}\eta^{(l)},$$

belongs to $\mathcal{T}_k^{\nu',\beta,+}$ while the polynomial

$$Z = \sum_{(j,l) \in \bar{\mathbb{N}}^k} c_{j,l} \xi^{(j)}\eta^{(l)}$$

belongs to $\mathcal{T}_k^{\nu',\beta}$ with $\nu' = \nu + \alpha$. Notice that in this non resonant case, (2.29) implies that Z depends only on the actions and thus is in normal form. Furthermore by construction they satisfy (2.28) and (2.29). Note that the reality of Q is equivalent to the symmetry relation: $\bar{a}_{jl} = a_{lj}$. Taking into account that $\Omega_{lj} = -\Omega_{jl}$, this symmetry remains satisfied for the polynomials χ and Z. Finally, χ and Z belong to $\mathcal{H}^s$, since they are homogeneous polynomials (they are their own Taylor expansions) and as a consequence of Proposition 2.13 (iii) and (iv) respectively. $\square$

We can now state the main result of this section:

Theorem 2.23. — *Assume that P is a real Hamiltonian belonging to $\mathcal{H}^s$ for all s large enough and to the class $\mathcal{T}^{\nu,\beta}$ for some $\nu \geq 0$ and $\beta > 0$. Assume that ω is strongly non resonant (cf. Definition 2.5) and satisfies* (2.1) *for some $\bar{d} \geq 0$. Then for any $r \geq 3$ there exists s_0 and for any $s \geq s_0$ there exists $\mathcal{U}_s$, $\mathcal{V}_s$ neighborhoods of the origin in $\mathcal{P}_s$ and $\tau_s : \mathcal{V}_s \to \mathcal{U}_s$ a real analytic canonical transformation which is the restriction to $\mathcal{V}_s$ of $\tau := \tau_{s_0}$ and which puts $H = H_0 + P$ in normal form up to order r i.e.*

$$H \circ \tau = H_0 + Z + R$$

with

(a) *Z is a real continuous polynomial of degree r with a regular vector field (i.e. $Z \in \mathcal{H}^s$) which only depends on the actions: $Z = Z(I)$.*
(b) *$R \in \mathcal{H}^s$ is real and $\|X_R(z)\|_s \leq C_s \|z\|_s^r$ for all $z \in \mathcal{V}_s$.*
(c) *τ is close to the identity: $\|\tau(z) - z\|_s \leq C_s \|z\|_s^2$ for all $z \in \mathcal{V}_s$.*

Proof. — The proof is close to the proof of Birkhoff normal form theorem stated in [**Gré07**] or [**Bam07**]. The main difference has been already pointed out : we have here to check the $\mathcal{H}^s$ regularity of the Hamiltonian functions at each step, independently of the fact that they belong to $\mathcal{T}^{\nu,\beta}$ (here $P \in \mathcal{T}^{\nu,\beta}$ does not imply $P \in \mathcal{H}^s$).
Having fixed some $r \geq 3$, the idea is to construct iteratively for $k = 3, \dots, r$, a neighborhood $\mathcal{V}_k$ of 0 in $\mathcal{P}_s$ (s large enough depending on r), a canonical transformation τ_k, defined on $\mathcal{V}_k$, an increasing sequence $(\nu_k)_{k=3,\dots,r}$ of positive numbers and real Hamiltonians $Z_k, P_{k+1}, Q_{k+2}, R_k$ such that

$$H_k := H \circ \tau_k = H_0 + Z_k + P_{k+1} + Q_{k+2} + R_k\,, \tag{2.32}$$

satisfying the following properties

(i) Z_k is a polynomial of degree k in $\mathcal{T}^{\nu_k,\beta} \cap \mathcal{H}^s$ having a zero of order 3 (at least) at the origin and Z_k depends only on the (new) actions: $\{Z_k, I_j\} = 0$ for all $j \geq 1$.
(ii) P_{k+1} is a homogeneous polynomial of degree $k+1$ in $\mathcal{T}^{\nu_k,\beta}_{k+1} \cap \mathcal{H}^s$.
(iii) Q_{k+2} is a polynomial of degree $r+1$ in $\mathcal{T}^{\nu_k,\beta} \cap \mathcal{H}^s$ having a zero of order $k+2$ at the origin.
(iv) R_k is a regular Hamiltonian belonging to $\mathcal{H}^s$ and having a zero of order $r+2$ at the origin.

First we fix $s > \nu_r + 3/2$ to be sure to be able to apply Proposition 2.13 at each step (ν_r will be defined later on independently of s). Then we notice that (2.32) at order r proves Theorem 2.23 with $Z = Z_r$ and $R = P_{r+1} + R_r$ (since $Q_{r+2} = 0$). Actually, since $R = P_{r+1} + R_r$ belongs to $\mathcal{H}^s$ and has a zero of order $r+1$ at the origin, we can apply Lemma 2.3 to obtain

$$\|X_R(z)\|_s \leq C_s \|z\|_s^r . \tag{2.33}$$

on $\mathcal{V} \subset \mathcal{V}_r$ a neighborhood of 0 in $\mathcal{P}_s$.
At the initial step (which for convenience we will denote the $k = 2$ step), the Hamiltonian $H = H_0 + P$ has the desired form (2.32) with $\tau_2 = I$, $\nu_2 = \nu$, $Z_2 = 0$, P_3 being the Taylor polynomial of P of degree 3, Q_4 being the Taylor polynomial of P of degree $r+1$ minus P_3 and $R_2 = P - P_3 - Q_4$. We show now how to go from step k to step $k+1$.
We look for τ_{k+1} of the form $\tau_k \circ \phi_{k+1}$, ϕ_{k+1} being the Lie transform associated to a homogeneous polynomial χ_{k+1} of degree $k+1$.

We decompose $H_k \circ \phi_{k+1}$ as follows

$$H_k \circ \phi_{k+1} = H_0 + Z_k + \{H_0, \chi_{k+1}\} + P_{k+1} \tag{2.34}$$
$$+ H_0 \circ \phi_{k+1} - H_0 - \{H_0, \chi_{k+1}\} \tag{2.35}$$
$$+ Z_k \circ \phi_{k+1} - Z_k \tag{2.36}$$
$$+ P_{k+1} \circ \phi_{k+1} - P_{k+1} \tag{2.37}$$
$$+ Q_{k+2} \circ \phi_{k+1} \tag{2.38}$$
$$+ R_k \circ \phi_{k+1} . \tag{2.39}$$

Using Lemma 2.22 above, we choose χ_{k+1} in $\mathcal{T}^{\nu'_k,\beta,+}_{k+1}$ in such a way that

$$\hat{Z}_{k+1} := \{H_0, \chi_{k+1}\} + P_{k+1} \tag{2.40}$$

is a homogeneous real polynomial of degree $k+1$ in $\mathcal{T}^{\nu'_k,\beta}_{k+1}$. We put then $Z_{k+1} = Z_k + \hat{Z}_{k+1}$, which obviously has degree $k+1$ and a zero of order 3 (at least) at the origin, and the right hand side of line (2.34) becomes $H_0 + Z_{k+1}$.

We just recall that $\nu'_k = \nu_k + \alpha$, where α is determined by ω, independently of r and s. By Proposition 2.21, the Lie transform associated to χ_{k+1} is well defined and smooth on a neighborhood $\mathcal{V}_{k+1} \subset \mathcal{V}_k$ and, for $z \in \mathcal{V}_{k+1}$ satisfies

$$\|\phi_{k+1}(z) - z\|_s \leq C \|z\|_s^2 .$$

Then from Proposition 2.18, Proposition 2.21 and formula (2.27), we find that (2.36), (2.37), (2.38) and (2.39) are regular Hamiltonians having zeros of order $k+2$ at the origin. For instance concerning (2.36), one has by Taylor formula for any $z \in \mathcal{V}_{k+1}$

$$Z_k \circ \phi_{k+1}(z) - Z_k(z) = \{Z_k, \chi_{k+1}\}(z) + \int_0^1 (1-t)\{\{Z_k, \chi_{k+1}\}, \chi_{k+1}\}(\Phi^t_{\chi_{k+1}}(z))\, dt$$

and $\{Z_k, \chi_{k+1}\}$ is a polynomial having a zero of order $3 + \text{degree}(\chi_{k+1}) - 2 = k+2$ while the integral term is a regular Hamiltonian having a zero of order $2k+1$. Thus if $2k+1 \geq r+2$ this last term contributes to R_{k+1} and if not, we have to use a Taylor formula at a higher order.
Therefore the sum of (2.36), (2.37), (2.38) and (2.39) decomposes in $\tilde{P}_{k+2} + \tilde{Q}_{k+3} + \tilde{R}_{k+1}$ with $\tilde{P}_{k+2}$, $\tilde{Q}_{k+3}$ and $\tilde{R}_{k+1}$ satisfying respectively the properties (ii), (iii) and (iv) at rank $k+1$ (with $\nu_{k+1} = k\nu'_k + \nu_k + k + 2$).
Concerning the term (2.35), one has to proceed differently since H_0 does not belong to the $\mathcal{H}^s$.
First notice that by the homological equation (2.40) one has $\{H_0, \chi_{k+1}\} = Z_{k+1} - Z_k - P_{k+1}$. By construction Z_k and P_{k+1} belong to $\mathcal{H}^s$. On the other hand, by Lemma 2.22, $Z_{k+1} \in \mathcal{T}^{\nu'_k, \beta}_{k+1}$ and is in normal form (i.e. it depends only on the action variables). Thus by Proposition 2.13 assertion (iv), one concludes that $Z_{k+1} \in \mathcal{H}^s$. Therefore we have proved that $\{H_0, \chi_{k+1}\} \in \mathcal{H}^s$.
Now we use the Taylor formula at order one to get

$$H_0 \circ \phi_{k+1}(z) - H_0(z) = \int_0^1 \{H_0, \chi_{k+1}\}(\Phi^t_{\chi_{k+1}}(z))\, dt.$$

But we know from the proof of Proposition 2.21 that $\Phi^t_{\chi_{k+1}} : \mathcal{V}_{k+1} \to \mathcal{P}_s$ for all $t \in [0,1]$. Therefore $H_0 \circ \phi_{k+1} - H_0 \in \mathcal{H}^s$ and thus (2.35) defines a regular Hamiltonian.
Finally we use again the Taylor formula and the homological equation to write

$$H_0 \circ \phi_{k+1}(z) - H_0(z) - \{H_0, \chi_{k+1}\}(z) = \int_0^1 (1-t)\{Z_{k+1} - Z_k - P_{k+1}, \chi_{k+1}\}(\Phi^t_{\chi_{k+1}}(z))\, dt$$

and , since $Z_{k+1} - Z_k - P_{k+1}$ belongs to $\mathcal{T}^{\nu'_k,\beta}_{k+1}$ and $\chi_{k+1} \in \mathcal{T}^{\nu'_k,\beta,+}_{k+1}$ we conclude by Proposition 2.18 that $H_0 \circ \phi_{k+1} - H_0 - \{H_0, \chi_{k+1}\} \in \mathcal{T}^{\nu_{k+1},\beta}$. Finally we use Proposition 2.21 to decompose it in $\hat{P}_{k+2} + \hat{Q}_{k+3} + \hat{R}_{k+1}$ with $\hat{P}_{k+2}$, $\hat{Q}_{k+3}$ and $\hat{R}_{k+1}$ satisfying respectively the properties (ii), (iii) and (iv) at rank $k+1$. The proof is achieved defining $P_{k+2} = \hat{P}_{k+2} + \tilde{P}_{k+2}$, $Q_{k+3} = \hat{Q}_{k+3} + \tilde{Q}_{k+3}$ and $R_{k+1} = \hat{R}_{k+1} + \tilde{R}_{k+1}$. $\square$

3. Dynamical consequences

3.1. Nonlinear harmonic oscillator in one dimension. — We recall the notations of the introduction. The quantum harmonic oscillator $T = -\frac{d^2}{dx^2} + x^2$ is diagonalized in the Hermite basis $(\phi_j)_{j\in\bar{\mathbb{N}}}$:

$$T\phi_j = (2j-1)\phi_j, \quad j \in \bar{\mathbb{N}}$$
$$\phi_{n+1} = \frac{H_n(x)}{\sqrt{2^n n!}} e^{-x^2/2}, \quad n \in \mathbb{N}$$

where $H_n(x)$ is the n^{th} Hermite polynomial relative to the weight e^{-x^2} :

$$\int_{\mathbb{R}} e^{-x^2} H_m(x) H_n(x) dx = 2^n n! \sqrt{\pi} \delta_{nm}.$$

In this basis, the Hermite multiplier is given by

$$M\phi_j = m_j \phi_j \tag{3.1}$$

where $(m_j)_{j\in\bar{\mathbb{N}}}$ is a bounded sequence of real number. For any $k \geq 1$, we define the class

$$\mathcal{W}_k = \{(m_j)_{j\in\bar{\mathbb{N}}} \mid \text{ for each } j,\ m_j = \frac{\tilde{m}_j}{j^k} \text{ with } \tilde{m}_j \in [-1/2, 1/2]\} \tag{3.2}$$

that we endow with the product probability measure. In this context the frequencies, i.e. the eigenvalues of $T + M = -d^2/dx^2 + x^2 + M$ are given by

$$\omega_j = 2j - 1 + m_j = 2j - 1 + \frac{\tilde{m}_j}{j^k}, \quad j \in \bar{\mathbb{N}}.$$

Proposition 3.1. — *There exists a set $F_k \subset \mathcal{W}_k$ whose measure equals 1 such that if $m = (m_j)_{j\in\bar{\mathbb{N}}} \in F_k$ then the frequency vector $(\omega_j)_{j\geq 1}$ is strongly non-resonant (cf. Definition 2.5).*

Proof. — First remark that it suffices to prove that the frequency vector $(\omega_j)_{j\geq 1}$ is non resonant in the sense of Definition 2.4. Actually, if we prove that (2.3) is satisfied for given constants δ' and γ' then if $S(j) < r\mu(j)$

$$\left|\omega_{j_1} + \cdots + \omega_{j_i} - \omega_{j_{i+1}} - \cdots - \omega_{j_r}\right| \geq \frac{\gamma'}{\mu(j)^{\delta'}} \geq \frac{\gamma'}{r+1}\frac{1+S(j)}{\mu(j)^{\delta'+1}}$$

and thus (2.4) is satisfied with $\delta = \delta' + 1$ and $\gamma = \frac{\gamma'}{r+1}$. Now if $S(j) \geq r\mu(j)$ then use

$$\left|\omega_{j_1} + \cdots + \omega_{j_i} - \omega_{j_{i+1}} - \cdots - \omega_{j_r}\right| \geq S(j) - (r-2)\mu(j), \tag{3.3}$$

to conclude that

$$\left|\omega_{j_1} + \cdots + \omega_{j_i} - \omega_{j_{i+1}} - \cdots - \omega_{j_r}\right| \geq \frac{2}{r}S(j) \geq \frac{\gamma'}{r+1} \quad \frac{1+S(j)}{\mu(j)^{\delta'+1}}$$

provided γ' is small enough.
The proof that there exists a set $F_k \subset \mathcal{W}_k$ whose measure equals 1 such that if $m = (m_j)_{j\in\bar{\mathbb{N}}} \in F_k$ then the frequency vector $(\omega_j)_{j\geq 1}$ is non resonant is exactly the same as the proof of Theorem 5.7 in [**Gré07**]. So we do not repeat it here (see also [**BG06**]). □

In equation (1.1) with $d = 1$, the Hamiltonian perturbation reads

$$P(\xi,\eta) = \int_{\mathbb{R}} g(\xi(x),\eta(x))dx \tag{3.4}$$

where $g \in C^\infty(\mathbb{C}^2,\mathbb{C})$, $\xi(x) = \sum_{j\geq 1}\xi_j\phi_j(x)$, $\eta(x) = \sum_{j\geq 1}\eta_j\phi_j(x)$ and $((\xi_j)_{j\geq 1},(\eta_j)_{j\geq 1}) \in \mathcal{P}_s$. We first check that P belongs to $\mathcal{H}^s$ for s large enough.

Lemma 3.2. — *Let P given by (3.4) with $g \in C^\infty(\mathcal{U},\mathbb{C})$, $\mathcal{U}$ being a neighborhood of 0 in $\mathbb{C}^2$, g real i.e. $g(z,\bar{z}) \in \mathbb{R}$ and g having a zero of order at least 3 at the origin. Then $P \in \mathcal{H}^s$ for all $s > 1/2$.*

Proof. — One computes

$$\frac{\partial P}{\partial \xi_j}(\xi,\eta) = \int_{\mathbb{R}} \partial_1 g(\xi(x),\eta(x))\phi_j(x)dx$$

and

$$\frac{\partial P}{\partial \eta_j}(\xi,\eta) = \int_{\mathbb{R}} \partial_2 g(\xi(x),\eta(x))\phi_j(x)dx\,.$$

In the same way, we have

$$\begin{aligned}
&\frac{\partial^{l+r} P}{\partial \xi_{j_1}\ldots\partial\xi_{j_l}\partial\eta_{k_1}\ldots\partial\eta_{k_r}}(\xi,\eta)\\
&= \int_{\mathbb{R}} \partial_1^l\partial_2^r g(\xi(x),\eta(x))\phi_{j_1}(x)\ldots\phi_{j_l}(x)\phi_{k_1}(x)\ldots\phi_{k_r}(x)dx\,. \qquad (3.5)
\end{aligned}$$

Since g is a C^∞ function, all these partial derivatives are continuous from $\mathcal{P}_s$ to $\mathbb{C}$, and the corresponding differentials $(\xi,\eta) \to D^{l+r}P(\xi,\eta)$ are continuous from $\mathcal{P}_s$ to the space of $l+r$-linear forms on $\mathcal{P}_s$. We get moreover

$$\|X_P(\xi,\eta)\|_s^2 = \sum_{j\geq 1} |j|^{2s} \left| \int_{\mathbb{R}} \partial_1 g(\xi(x),\eta(x))\phi_j(x)dx \right|^2 + \sum_{j\geq 1} |j|^{2s} \left| \int_{\mathbb{R}} \partial_2 g(\xi(x),\eta(x))\phi_j(x)dx \right|^2 .$$

Therefore, to check that $z \mapsto X_P(z)$ is a regular function from a neighborhood of the origin in $\mathcal{P}_s$ into $\mathcal{P}_s$, it suffices to check that the functions $x \mapsto \partial_1 g(\xi(x),\eta(x))$ and $x \mapsto \partial_2 g(\xi(x),\eta(x))$ are in $\tilde{H}^s$ provided $\xi(x)$ and $\eta(x)$ are in $\tilde{H}^s$. So it remains to prove that functions of the type $x \mapsto |x|^i \partial_1^{l+1}\partial_2^m g(\xi(x),\eta(x))(\xi^{(l_1)}(x))^{\alpha_1}\cdot(\xi^{(l_{k_1})}(x))^{\alpha_{k_1}}(\eta^{(m_1)}(x))^{\beta_1}\cdots(\eta^{(m_{k_2})}(x))^{\beta_{k_2}}$ are in $L^2(\mathbb{R})$ for all $0\leq i+l+m\leq s$, $0\leq i+l_j\leq s$, $0\leq i+m_j\leq s$. But this is true because

- g is a C^∞ function, ξ and η are bounded functions and thus $x \mapsto \partial_1^{l+1}\partial_2^m g(\xi(x),\eta(x))$ is bounded
- $\tilde{H}^s$ is an algebra for $s>1/2$ and thus $x \mapsto |x|^k \xi^{(l)}(x)\eta^{(m)}(x) \in L^2(\mathbb{R})$ for all $0\leq k+l+m\leq s$.
- $|\partial_1 g(\xi(x),\eta(x))|$, $|\partial_2 g(\xi(x),\eta(x))| \leq C(|\xi(x)|+|\eta(x)|)^2$ for some uniform constant $C>0$ and thus $x \mapsto |x|^k \partial_1 g(\xi(x),\eta(x)) \in L^2(\mathbb{R})$ for all $0\leq k\leq s$.

There remains to prove the same properties concerning the Taylor homogeneous polynomial P_m of P at any order m, computed at $(0,0)$. From (3.5), we get

$$P_m = \frac{1}{m!}\int_{\mathbb{R}} \sum_{l+r=m} \partial_1^l\partial_2^r g(0,0) \sum_{j,k} \xi_{j_1}\phi_{j_1}(x)\ldots\xi_{j_l}\phi_{j_l}(x)\eta_{k_1}\phi_{k_1}(x)\ldots\eta_{k_r}\phi_{k_r}(x)dx\,,$$

hence P_m can be computed directly from formula (3.4), replacing g by its Taylor homogeneous polynomial g_m of order m :

$$g_m(\xi(x),\eta(x)) = \frac{1}{m!}\sum_{l+r=m} \partial_1^l\partial_2^r g(0,0)\xi(x)^l\eta(x)^r,,$$

and this gives the statement, since g_m satisfies the same properties as g. □

The fact that P belongs to the class $\mathcal{T}^{\nu,\beta}$ is directly related to the distribution of the ϕ_j's. Actually we have

Proposition 3.3. — *Let $\nu > 1/8$ and $0 \le \beta \le \frac{1}{24}$. For each $k \ge 1$ and for each $N \ge 0$ there exists $c_N > 0$ such that for all $j \in \bar{\mathbb{N}}^k$*

$$\left| \int_{\mathbb{R}} \phi_{j_1} \dots \phi_{j_k} dx \right| \le c_N \frac{\mu(j)^\nu}{C(j)^\beta} A(j)^N. \tag{3.6}$$

As a consequence, any P of the general form (3.4) *is in the class $\mathcal{T}^\nu$.*

The proof will be done in the multidimensional case in the next section (cf. Proposition 3.6).

We can now apply our Theorem 2.23 to obtain

Theorem 3.4. — *Assume that $M \in F_m$ defined in Proposition 3.1 and that $g \in C^\infty(\mathbb{C}^2, \mathbb{C})$ is real i.e. $g(z, \bar{z}) \in \mathbb{R}$ and has a zero of order at least 3 at the origin. For any $r \ge 3$ there exists $s_0(r)$ an integer such that for any $s \ge s_0(r)$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that if $\|\psi_0\|_{\tilde{H}^{2s}} = \varepsilon < \varepsilon_0$ the equation*

$$i\psi_t = (-\Delta + x^2 + M)\psi + \partial_2 g(\psi, \bar{\psi}), \quad x \in \mathbb{R},\ t \in \mathbb{R} \tag{3.7}$$

with Cauchy data ψ_0 has a unique solution $\psi \in C^1((-T_\epsilon, T_\epsilon), \tilde{H}^{2s})$ with

$$T_\epsilon \ge C\epsilon^{-r}. \tag{3.8}$$

Moreover

(i) $\|\psi(t,\cdot)\|_{\tilde{H}^{2s}} \le 2\epsilon$ *for any $t \in (-T_\epsilon, T_\epsilon)$.*

(ii) $\sum_{j\ge1} j^{2s} ||\xi_j(t)|^2 - |\xi_j(0)|^2| \le \varepsilon^3$ *for any $t \in (-T_\epsilon, T_\epsilon)$ where $|\xi_j(t)|^2$, $j \ge 1$ are the actions of $\psi(t,\cdot) = \sum \xi_j(t)\phi_j$.*

(iii) *there exists a torus $\mathcal{T}_0 \subset \tilde{H}^{2s}$ such that,*

$$dist_{2s}(\psi(t,\cdot), \mathcal{T}_0) \le C\varepsilon^{r_1/2} \qquad \textit{for } |t| \le \epsilon^{-r_2}$$

where $r_1 + r_2 = r + 3$ and $dist_{2s}$ denotes the distance on $\tilde{H}^{2s}$ associated with the norm $\|\cdot\|_{\tilde{H}^{2s}}$.

Proof. — Let $\psi_0 = \sum \xi_j(0)\phi_j$ and denote $z_0 = (\xi(0), \bar{\xi}(0))$. Notice that if $\psi_0 \in \tilde{H}^{2s}$ with $\|\psi_0\|_{\tilde{H}^{2s}} = \epsilon$ then $z_0 \in \mathcal{P}_s$ and $\|z_0\|_s = \epsilon$. Denote by $z(t)$ the solution of the Cauchy problem $\dot{z} = X_H(z)$, $z(0) = z_0$, where $H = H_0 + P$ is the Hamiltonian function associated to the equation (3.7) written in the Hermite decomposition $\psi(t) = \sum \xi_j(t)\phi_j$, $z(t) = (\xi(t), \bar{\xi}(t))$. We note that, since P is real, z remains a real point of $\mathcal{P}_s$ for all t and that $\|\psi(t)\|_{\tilde{H}^{2s}} = \|z(t)\|\, s$.
Then we denote by $z' = \tau^{-1}(z)$ where $\tau : \mathcal{V}_s \to \mathcal{U}_s$ is the transformation given by Theorem 2.23 (so that z' denotes the normalized coordinates) associated to the order $r + 2$ and $s \ge s_0(r + 2)$ given by the same Theorem. We note that, since the transformation τ is generated by a real Hamiltonian, $z'(t)$ is still a real point.

Let $\varepsilon_0 > 0$ be such that $B_{2\varepsilon_0} \subset \mathcal{V}_s$ and take $0 < \varepsilon < \varepsilon_0$. We assume that $\|z(0)\|_s = \|\psi_0\|_{\tilde{H}^s} = \varepsilon$. For $z = (\xi, \eta) \in \mathcal{P}_s$ we define

$$N(z) := 2\sum_{j=1}^{\infty} j^{2s} I_j(\xi, \eta)$$

where we recall that $I_j(\xi, \eta) = \xi_j \eta_j$. We notice that for a real point $z = (\xi, \bar{\xi}) \in \mathcal{P}_s$,

$$N(z) = \|z\|_s^2 .$$

Thus in particular we have[5]

$$N(z(t)) = \|z(t)\|_s^2 \quad \text{ and } N(z'(t)) = \left\|z'(t)\right\|_s^2 .$$

Using that Z depends only on the normalized actions, we have

$$\dot{N}(z') = \{N, H \circ \tau\} \circ \tau^{-1}(z) = \{N, R\}(z'). \tag{3.9}$$

Therefore as far as $\|z(t)\|_s \leq 2\varepsilon$, and thus $z(t) \in \mathcal{V}_s$, by assertion (c) of Theorem 2.23, $\|z'(t)\|_s \leq C\varepsilon$ and using (3.9) and assertion (b) of Theorem 2.23 (at order $r + 2$) we get

$$\left|N(z'(t)) - N(z'(0))\right| \leq \left|\int_0^t \{N, R\}(z'(t'))dt'\right| \leq Ct \left\|z'(t)\right\|_s^{r+3} \leq Ct\varepsilon^{r+3}.$$

In particular, as far as $\|z(t)\|_s \leq 2\varepsilon$ and $|t| \leq C\varepsilon^{-r}$

$$\left|N(z'(t)) - N(z'(0))\right| \leq C\varepsilon^3.$$

Therefore using again assertion (c) of Theorem 2.23, we obtain

$$|N(z(t)) - N(z(0))| \leq C\varepsilon^3$$

which, choosing ε_0 small enough, leads to $\|z(t)\|_s \leq 3/2\, \varepsilon$ as long as $\|z(t)\|_s \leq 2\varepsilon$ and $|t| \leq C\varepsilon^{-r}$. Thus (3.8) and assertions (i) follow by a continuity argument.

To prove assertion (ii) we recall the notation $I_j(z) = I_j(\xi, \eta) = \xi_j \eta_j$ for the actions associated to $z = (\xi, \eta)$. Using that Z depends only on the actions, we have

$$\{I_j \circ \tau^{-1}, H\}(z) = \{I_j, H \circ \tau\} \circ \tau^{-1}(z) = \{I_j, R\}(z').$$

Therefore, we get in the normalized coordinates

$$\frac{d}{dt} I_j(\xi', \eta') = -i\xi'_j \frac{\partial R}{\partial \eta_j} + i\eta'_j \frac{\partial R}{\partial \xi_j}$$

[5] That is precisely at this point that we need to work with real Hamiltonians. The Birkhoff normal form theorem is essentially algebraic and does hold for complex Hamiltonians.

and thus

$$\sum_j j^{2s}\left|\frac{d}{dt}I_j(\xi',\bar\xi')\right| = \sum_j j^{2s}\left|-\xi'_j\frac{\partial R}{\partial \eta_j}+\bar\xi'_j\frac{\partial R}{\partial \xi_j}\right|$$
$$\leq \left(\sum_j j^{2s}(|\xi'_j|^2+|\xi'_j|^2)\right)^{1/2}\left(\sum_j j^{2s}\left(\left|\frac{\partial R}{\partial \eta_j}\right|^2+\left|\frac{\partial R}{\partial \xi_j}\right|^2\right)\right)^{1/2}$$

which leads to

$$\sum_j j^{2s}\left|\frac{d}{dt}I_j(z')\right| \leq \|z'\|_s\,\|X_R(z')\|_s \leq \|z'\|_s^{r+3}. \tag{3.10}$$

Thus, recalling that $I_j(\xi',\bar\xi')=|\xi'_j|^2$ we get

$$\sum_{j\geq 1} j^{2s}\left||\xi'_j(t)|^2-|\xi'_j(0)|^2\right| \leq \varepsilon^3 \text{ for any } |t|\leq C\varepsilon^{-r}. \tag{3.11}$$

On the other hand, using (i) and assertion (c) of Theorem 2.23, for any $|t|\leq C\varepsilon^{-r}$, one has

$$\sum_{j\geq 1} j^{2s}\left||\xi_j(t)|^2)-|\xi'_j(t)|^2\right| \leq \sum_{j\geq 1} j^{2s}(|\xi_j(t)|+|\xi'_j(t)|)\left|\xi_j(t)-\xi'_j(t)\right| \leq C\varepsilon^3.$$

Combining this last relation with (3.11), assertion (ii) follows.

To prove (iii), let $\bar I_j = I'_j(0)$ be the initial actions in the normalized coordinates and define the smooth torus

$$\Pi_0 := \left\{z\in\mathcal{P}_s \ : \ I_j(z)=\bar I_j\ , j\geq 1\right\}$$

and its image in $\tilde H^s$

$$\mathcal{T}_0=\{u\in\tilde H^s \ : \ u=\sum \xi_j\phi_j \text{ with } \tau(\xi,\bar\xi)\in\Pi_0\}.$$

We have

$$d_s(z(t),\mathcal{T}_0) \leq \left[\sum_j j^{2s}\left|\sqrt{I'_j(t)}-\sqrt{\bar I_j}\right|^2\right]^{1/2} \tag{3.12}$$

where d_s denotes the distance in $\mathcal{P}_s$ associated to $\|\cdot\|_s$.
Notice that for $a,b\geq 0$,

$$\left|\sqrt{a}-\sqrt{b}\right| \leq \sqrt{|a-b|}\ .$$

Thus, using (3.10), we get

$$\begin{aligned}[d_s(z(t),\mathcal{T}_0)]^2 &\leq \sum_j j^{2s}|I'_j(t)-I'_j(0)|\\ &\leq |t|\sum_j j^{2s}|\dot{I}'_j(t)|\\ &\leq \frac{1}{\epsilon^{r_1}}\left\|z'\right\|_s\left\|X_R(z')\right\|_s\\ &\leq C\frac{1}{\epsilon^{r_1}}\epsilon^{r+3}\leq C\epsilon^{r+3-r_1}.\end{aligned}$$

which gives (ii).

□

3.2. Multidimensional nonlinear harmonic oscillator. —

3.2.1. Model. — The spectrum of the d-dimensional harmonic oscillator

$$T=-\Delta+|x|^2=-\Delta+x_1^2+\cdots+x_d^2$$

is the sum of d-copies of the odd integers set, i.e. the spectrum of T equals $\mathbb{N}_d$ with

$$\mathbb{N}_d=\begin{cases}2\mathbb{N}\setminus\{0,2,\cdots,d-2\} \text{ if d is even}\\ 2\mathbb{N}+1\setminus\{1,3,\cdots,d-2\} \text{ if d is odd.}\end{cases} \tag{3.13}$$

For $j\in\mathbb{N}_d$ we denote the associated eigenspace E_j which dimension is

$$d_j=\sharp\{(i_1,\cdots,i_d)\in(2\mathbb{N}+1)^d \mid i_1+\cdots+i_d=j\}.$$

We denote $\{\Phi_{j,l},\ l=1,\cdots,d_j\}$, the basis of E_j obtained by d-tensor product of Hermite functions: $\Phi_{j,l}=\phi_{i_1}\otimes\cdots\phi_{i_d}$ with $i_1+\cdots+i_d=j$.

The Hermite multiplier M is defined on the basis $(\Phi_{j,l})_{j\in\mathbb{N}_d,l=1,\cdots,d_j}$ of $L^2(\mathbb{R}^d)$ by

$$M\Phi_{j,l}=m_{j,l}\Phi_{j,l} \tag{3.14}$$

where $(m_{j,l})_{j\in\mathbb{N}_d,l=1,\cdots,d_j}$ is a bounded sequence of real numbers.

The linear part of (1.1) reads

$$H_0=-\Delta+x^2+M.$$

H_0 is still diagonalized by $(\Phi_{j,l})_{j\in\mathbb{N}_d,l=1,\cdots,d_j}$ and the spectrum of H_0 is

$$\sigma(H_0)=\{j+m_{j,l}\mid j\in\mathbb{N}_d,l=1,\cdots,d_j\} \tag{3.15}$$

For simplicity, we will focus on the case $m_{j,l}=m_j$ for all $l=1,\cdots,d_j$. In this case we have $\sigma(H_0)=\{j+m_j\mid j\in\mathbb{N}_d\}$ and, as a consequence of Proposition 3.1,

Proposition 3.5. — *There exists a set $F_k \subset \mathcal{W}_k$ whose measure equals* 1 *such that if $m = (m_j)_{j\in\bar{\mathbb{N}}} \in F_k$ then the frequency vector $(\omega_{j,i})_{j\in\mathbb{N}_d, i=1,\cdots,d_j}$ satisfies the following:*
for any $r \in \bar{\mathbb{N}}$, there are $\gamma > 0$ and $\delta > 0$ such that for any $j \in {\mathbb{N}_d}^r$, any $l \in \{1, \cdots, d_{j_1}\} \times \cdots \times \{1, \cdots, d_{j_r}\}$ and any $1 \leq i \leq r$, one has

$$\left|\omega_{j_1,l_1} + \cdots + \omega_{j_i,l_i} - \omega_{j_{i+1},l_{i+1}} - \cdots - \omega_{j_r,l_r}\right| \geq \gamma \frac{1 + S(j)}{\mu(j)^\delta} \tag{3.16}$$

except if $\{j_1, \ldots, j_i\} = \{j_{i+1}, \ldots, j_r\}$.

Concerning the product of eigenfunctions we have,

Proposition 3.6. — *Let $\nu > d/8$. For any $k \geq 1$ and any $N \geq 1$ there exists $c_N > 0$ such that for any $j \in {\mathbb{N}_d}^k$, any $l \in \{1, \cdots, d_{j_1}\} \times \cdots \times \{1, \cdots, d_{j_k}\}$*

$$\left|\int_{\mathbb{R}^d} \Phi_{j_1,l_1} \ldots \Phi_{j_k,l_k} dx\right| \leq c_N \frac{\mu(j)^\nu}{C(j)^{\frac{1}{24}}} A(j)^N. \tag{3.17}$$

Notice that this condition does not distinguish between modes having the same energy.

Proof. — We use the approach developed in [**Bam07**] Section 6.2. The basic idea lies in the following commutator lemma: Let A be a linear operator which maps $D(T^k)$ into itself and define the sequence of operators

$$A_N := [T, A_{N-1}], \quad A_0 := A$$

then ([**Bam07**] Lemma 7) for any $j_1 \neq j_2$ in $\mathbb{N}_d$, any $0 \leq l_1 \leq d_1$, $0 \leq l_2 \leq d_2$ and any $N \geq 0$

$$|\langle A\Phi_{j_2,l_2}, \Phi_{j_1,l_1}\rangle| \leq \frac{1}{|j_1 - j_2|^N} |\langle A_N \Phi_{j_2,l_2}, \Phi_{j_1,l_1}\rangle|.$$

Let A be the operator given by the multiplication by the function $\Phi = \Phi_{j_3,l_3} \cdots \Phi_{j_k,l_k}$ then by an induction argument

$$A_N = \sum_{0\leq|\alpha|\leq N} C_{\alpha,N} D^\alpha$$

where

$$C_{\alpha,N} = \sum_{0\leq|\beta|\leq 2N-|\alpha|} V_{\alpha,\beta,N}(x) D^\beta \phi$$

and $V_{\alpha,\beta,N}$ are polynomials of degree less than $2N$. Therefore one gets

(3.18)
$$\begin{aligned}\left|\int_{\mathbb{R}^d}\Phi_{j_1,l_1}\dots\Phi_{j_k,l_k}dx\right| &\leq \frac{1}{|j_1-j_2|^N}\left\|A_N\Phi_{j_2,l_2}\right\|_{L^2}\\ &\leq C\frac{1}{|j_1-j_2|^N}\sum_{0\leq|\alpha|\leq N}\sum_{0\leq|\beta|\leq 2N-|\alpha|}||V_{\alpha,\beta,N}D^\beta\phi D^\alpha\Phi_{j_2,l_2}||_{L^2}\\ &\leq C\frac{1}{|j_1-j_2|^N}\sum_{0\leq|\alpha|\leq N}\sum_{0\leq|\beta|\leq 2N-|\alpha|}\left\|\Phi_{j_2,l_2}\right\|_{|\alpha|}||\Phi||_{\nu_0+|\beta|}\end{aligned}$$

where we used in the last estimate (in this proof, $\|f\|_s = \|f\|_{H^s(\mathbb{R}^d)}$, the standard Sobolev norm)

$$\forall\nu_0>d/2 \qquad ||fg||_{L^2}\leq C_{\nu_0}||f||_{\nu_0}||g||_{L^2}.$$

We now estimate $||\Phi||_{\nu_0+|\beta|}$. First notice that, since $T\Phi_{j,l}=j\Phi_{j,l}$, one has for all $s\geq 0$

$$\left\|\Phi_{j,l}\right\|_s\leq Cj^{s/2}. \tag{3.19}$$

Then we recall that the Hermite eigenfunctions are uniformly bounded, and in fact (see [**Sze75**] or [**KT05**])

$$||\phi_j||_{L^\infty}\leq Cj^{-1/12}, \tag{3.20}$$

and thus, since $\Phi_{j,l}=\phi_{i_1}\otimes\cdots\phi_{i_d}$ with $i_1+\cdots+i_d=j$, we deduce

$$||\Phi_{j,l}||_{L^\infty}\leq C_d j^{-1/12} \tag{3.21}$$

with $C_d=Cd^{1/12}$. Thus using tame estimates (see for instance [**Tay91**])

$$\|uv\|_s\leq C(\|u\|_s\|v\|_{L^\infty}+\|v\|_s\|u\|_{L^\infty})$$

combined with (3.19) and (3.21), we get for $j_3\geq\cdots\geq j_k$,

$$\|\Phi\|_s\leq Cj_3^{s/2}. \tag{3.22}$$

Inserting (3.19) and (3.22) in (3.18) we get

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\Phi_{j_1,l_1}\dots\Phi_{j_k,l_k}dx\right| &\leq C\frac{1}{|j_1-j_2|^N}\sum_{0\leq|\alpha|\leq N}\sum_{0\leq|\beta|\leq 2N-|\alpha|} j_2^{|\alpha|/2}j_3^{(\nu_0+|\beta|)/2)}\\ &\leq C\frac{1}{|j_1-j_2|^N}\sum_{0\leq|\alpha|\leq N} j_2^{|\alpha|/2}j_3^{\nu_0/2+N-|\alpha|/2}\\ &\leq C\frac{1}{|j_1-j_2|^N}j_3^{N+\nu_0/2}\left(\frac{j_2}{j_3}\right)^{N/2}\\ &= C\frac{1}{|j_1-j_2|^N}j_3^{\nu_0/2}(j_2j_3)^{N/2}.\end{aligned}$$

Now remark that if $\sqrt{j_2j_3}\leq|j_1-j_2|$ then the last estimate implies

(3.23)
$$\left|\int_{\mathbb{R}^d}\Phi_{j_1,l_1}\dots\Phi_{j_k,l_k}dx\right|\leq Cj_3^{\nu_0/2}\frac{(j_2j_3)^{N/2}}{(\sqrt{j_2j_3}+|j_1-j_2|)^N}=C\mu(j)^{\nu/2}A(j)^N$$

while if $\sqrt{j_2j_3}>|j_1-j_2|$ then $A(j)\geq 1/2$ and thus (3.23) is trivially true.

On the other hand, using (3.21) one has

$$\left|\int_{\mathbb{R}^d}\Phi_{j_1,l_1}\dots\Phi_{j_k,l_k}dx\right|\leq Cj_1^{-1/12}=C\ C(j)^{-1/12}.$$

Combining this estimate with (3.23) one gets for all $N\geq 1$

$$\left|\int_{\mathbb{R}^d}\Phi_{j_1,l_1}\dots\Phi_{j_k,l_k}dx\right|\leq c_N\frac{\mu(j)^\nu}{C(j)^{\frac{1}{24}}}A(j)^N$$

with $\nu=\frac{\nu_0}{4}$. □

3.2.2. Result. — We first generalize the normal form theorem to a context adapted to the multidimensional case. We follow the presentation of Section 2 and only focus on the new features.

Let $s\geq 0$, we consider the phase space $\mathcal{Q}_s=\mathcal{L}_s\times\mathcal{L}_s$ with

$$\mathcal{L}_s=\{(a_{j,l})_{j\in\mathbb{N}_d,\,1\leq l\leq d_j}\mid\sum_{j\in\mathbb{N}_d}|j|^{2s}\sum_{l=1}^{d_j}|a_{j,l}|^2<\infty\}$$

that we endow with the standard norm and the standard symplectic structure as for $\mathcal{P}_s$ in Section 2.1. Writing $\psi=\sum\xi_{j,l}\Phi_{j,l}$, $\bar\psi=\sum\eta_{j,l}\Phi_{j,l}$ with $(\xi,\eta)\in\mathcal{Q}_s$, we note that $\psi\in\tilde H^{2s}$ if and only if $\xi\in\mathcal{L}_s$. The linear part of the multidimensional version of the linear part of (1.1) reads

$$H_0(\xi,\eta)=\frac{1}{2}\sum_{j\in\mathbb{N}_d}\sum_{l=1}^{d_j}\omega_{j,l}\xi_{j,l}\eta_{j,l}.$$

For $j\geq 1$, we define

$$J_j(\xi,\eta)=\sum_{l=1}^{d_j}\xi_{j,l}\eta_{j,l}\ .$$

Using notations of Section 2.1, we define the class $\mathbb{T}_k^\nu$ of homogeneous polynomials of degree k on $\mathcal{Q}_s$

$$Q(\xi,\eta)\equiv Q(z)=\sum_{j\in{\mathbb{N}_d}^k}\sum_{l_1=1}^{d_{j_1}}\dots\sum_{l_m=1}^{d_{j_k}}a_{j,l}z_{j_1,l_1}\dots z_{j_k,l_k}$$

such that for each $N\geq 1$, there exists a constant $C>0$ such that for all j,l

$$|a_{j,l}|\leq C\frac{\mu(j)^\nu}{C(j)^{1/24}}A(j)^N.$$

Then, following Definition 2.11 we define a corresponding class $\mathbb{T}^\nu$ of C^∞ Hamiltonians on $\mathcal{Q}_s$ having their Taylor polynomials in $\mathbb{T}_k^\nu$. Similarly, following Definition 2.1, we also define $\mathcal{H}_d^s$ the class of real Hamiltonians P satisfying $P,P_k\in C^\infty(\mathcal{U}_s,\mathbb{C})$ and $X_P,X_{P_k}\in C^\infty(\mathcal{U}_s,\mathcal{Q}_s)$ for some $\mathcal{U}_s\subset\mathcal{Q}_s$ a neighborhood of the origin and for all $k\geq 1$ (as before P_k denotes the Taylor polynomial of P of degree k).
In equation (1.1), the Hamiltonian perturbation reads

$$P(\xi,\eta)=\int_{\mathbb{R}^d}g(\xi(x),\eta(x))dx \tag{3.24}$$

where g is C^∞ on a neighborhood of 0 in $\mathbb{C}^2$, $\xi(x)=\sum_{j\geq1}\xi_j\phi_j(x)$, $\eta(x)=\sum_{j\geq1}\eta_j\phi_j(x)$ and $((\xi_j)_{j\geq1},(\eta_j)_{j\geq1})\in\mathcal{P}_s$. As in the one dimensional case (cf. Lemma 3.2), P belongs to $\mathcal{H}_d^s$ for s large enough ($s>d/2$) and using Proposition 3.6, P belongs to the class $\mathbb{T}^\nu$. Therefore one has

Lemma 3.7. — *Let P given by* (3.24) *with g smooth, real and having a zero of order at least 3 at the origin. Then $P\in\mathcal{H}^s\cap\mathbb{T}^\nu$ for all $s>d/2$ and for $\nu>d/8$.*

We also need a d-dimensional definition of *normal form* homogeneous polynomial :

Definition 3.8. — *Let $k = 2m$ be an even integer, a formal polynomial Z homogeneous of degree k on $\mathcal{Q}_s$ is in normal form if it reads*

$$Z(\xi,\eta) = \sum_{j\in\mathbb{N}_d{}^k} \sum_{l_1,l_1'=1}^{d_{j_1}} \cdots \sum_{l_k,l_k'=1}^{d_{j_k}} a_{j,l,l'}\xi_{j_1,l_1}\eta_{j_1,l_1'}\dots\xi_{j_k,l_k}\eta_{j_k,l_k'}$$

for all $(\xi,\eta)\in\mathcal{Q}_s$.

One easily verifies that if Z is in normal form then Z commutes with each $J_j = \sum_{l=1}^{d_j}\xi_{j,l}\eta_{j,l}$ since for instance

$$\{\xi_{j_1,l_1}\eta_{j_1,l_1'},\xi_{j_1,l_1}\eta_{j_1,l_1}+\xi_{j_1,l_1'}\eta_{j_1,l_1'}\}=0.$$

Modifying slightly the proof of Theorem 2.23 we get

Theorem 3.9. — *Let P be a real Hamiltonian belonging in $\mathbb{T}^\nu\cap\mathcal{H}_d^s$ for some $\nu\geq 0$ and for all s sufficiently large and let ω be a weakly non resonant frequency vector in the sense of* (3.16)*. Then for any $r\geq 3$ there exists s_0 and for any $s\geq s_0$ there exists $\mathcal{U}$, $\mathcal{V}$ neighborhoods of the origin in $\mathcal{Q}_s$ and $\tau:\mathcal{V}\to\mathcal{U}$ a real analytic canonical transformation which puts $H=H_0+P$ in normal form up to order r i.e.*

$$H\circ\tau = H_0+Z+R$$

with

(i) *Z is a real continuous polynomial of degree r which belongs to $\mathcal{H}_d^s$ and which is in normal form in the sense of Definition 3.8. In particular Z commutes with all J_j, $j\geq 1$, i.e. $\{Z,J_j\}=0$ for all $j\geq 1$.*

(ii) *R is real and belongs to $\mathcal{H}_d^s$, furthermore $\|X_R(z)\|_s\leq C_s\|z\|_s^r$ for all $z\in\mathcal{V}_s$.*

(iii) *τ is close to the identity: $\|\tau(z)-z\|_s\leq C_s\|z\|_s^2$ for all $z\in\mathcal{V}_s$.*

Proof. — The only new point when comparing with Theorem 2.23, is that in assertion (ii) we obtain $\{Z,J_j\}=0$ for all $j\geq 1$ instead of $\{Z,I_j\}=0$ for all $j\geq 1$. Actually, in view of (3.16), we adapt Lemma 2.22, and in particular (2.30) and (2.31), in such a way $\chi\in\mathbb{T}^{\nu,+}$ and Z is in normal form in the sense of Definition 3.8.

On the other hand, we also verify, following the lines of the proof of assertion (iv) of Proposition 2.13, that a homogeneous polynomial of degree $k+1$ in normal form $Z\in\mathbb{T}^\nu$ satisfies $\|X_Z(z)\|_s\leq C\|z\|_s^k$ for all z in a neighborhood of the origin. In particular, if $Z\in\mathbb{T}^\nu$ is in normal form, it automatically belongs to $\mathcal{H}_d^s$ (this point was crucial in the proof of Theorem 2.23).

□

Notice that the normal form $H_0 + Z$ is no longer, in general, integrable. The dynamical consequences are the same as in Theorem 3.4 (i) and (ii) but we have to replace I_j by J_j in the second assertion. Actually the J_j play the rôle of almost actions: they are almost conserved quantities.

Theorem 3.10. — *Assume that $m \in F_k$ defined in proposition 3.5 and that g is C^∞ on a neighborhood of 0 in $\mathbb{C}^2$, g is real i.e. $g(z,\bar{z}) \in \mathbb{R}$ and g vanishes at least at order 3 at the origin. For each $r \geq 3$ and $s \geq s_0(r)$, there exists $\varepsilon_0 > 0$ and $c > 0$ such that for any ψ_0 in $\tilde{H}^s$, any $\epsilon \in (0, \epsilon_0)$, the equation*

$$i\psi_t = (-\Delta + x^2 + M)\psi + \partial_2 g(\psi, \bar{\psi}), \quad x \in \mathbb{R}^d,\ t \in \mathbb{R}$$

with Cauchy data ψ_0 has a unique solution $\psi \in C^1((-T_\epsilon, T_\epsilon), \tilde{H}^s)$ with

$$T_\epsilon \geq c\epsilon^{-r}.$$

Moreover for any $t \in (-T_\epsilon, T_\epsilon)$, one has

$$\|\psi(t,\cdot)\|_{\tilde{H}^s} \leq 2\epsilon$$

and

$$\sum_{j\geq 1} j^s |J_j(t) - J_j(0)| \leq \varepsilon^3$$

where $J_j(t) = \sum_{l=1}^{d_j} |\xi_{j,l}|^2$, $j \geq 1$ are the "pseudo-actions" of $\psi(t,\cdot) = \sum_{j,l} \xi_{j,l}(t)\Phi_{j,l}(\cdot)$.

Proof. — Just remark that as in the proof of Theorem 3.4, defining $N(z) := 2\sum_{j\in\mathbb{N}_d} j^s J_j(\xi,\eta) = 2\sum_{j\in\mathbb{N}_d} j^s \sum_{l=1}^{d_j} \xi_{j,l}\eta_{j,l}$ one has $N(z) = \|z\|_{s/2}^2$ for all real point $z = (\xi, \bar{\xi})$. On the other hand, using that Z commutes with J_j, we have

$$\{N \circ \tau^{-1}, H\}(z) = \{N, H \circ \tau\} \circ \tau^{-1}(z) = \{N, R\}(z').$$

Therefore, in the normalized variables, we have the estimate $|\dot{N}| \leq CN^{(r+1)/2}$ and the theorem follows as in the proof of Theorem 3.4.

$\square$

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